CHAPTER 1

Functions and Graphs

One of the central principles of economics is that the value of money is not constant; it is a function of time. Since fortunes are made and lost by people attempting to predict the future value of money, much attention is paid to quantitative measures like the Consumer Price Index, a basic measure of inflation in various sectors of the economy. See page 146 for a look at how the Consumer Price Index for housing has behaved over time.

Chapter 1 Overview

In this chapter we begin the study of functions that will continue throughout the book. Your previous courses have introduced you to some basic functions. These functions can be visualized using a graphing calculator, and their properties can be described using the notation and terminology that will be introduced in this chapter. A familiarity with this terminology will serve you well in later chapters when we explore properties of functions in greater depth.

1.1 Modeling and Equation Solving

What you'll learn about

- Numerical Models
- Algebraic Models
- Graphical Models
- The Zero Factor Property
- Problem Solving
- Grapher Failure and Hidden Behavior
- A Word About Proof

... and why

Numerical, algebraic, and graphical models provide different methods to visualize, analyze, and understand data.

Numerical Models

Scientists and engineers have always used mathematics to model the real world and thereby to unravel its mysteries. A **mathematical model** is a mathematical structure that approximates phenomena for the purpose of studying or predicting their behavior. Thanks to advances in computer technology, the process of devising mathematical models is now a rich field of study itself, **mathematical modeling**.

We will be concerned primarily with three types of mathematical models in this book: *numerical models*, *algebraic models*, and *graphical models*. Each type of model gives insight into real-world problems, but the best insights are often gained by switching from one kind of model to another. Developing the ability to do that will be one of the goals of this course.

Perhaps the most basic kind of mathematical model is the **numerical model**, in which numbers (or *data*) are analyzed to gain insights into phenomena. A numerical model can be as simple as the major league baseball standings or as complicated as the network of interrelated numbers that measure the global economy.

Table 1.1 The Minimum Hourly Wage

Year	Minimum Hourly Wage (MHW)	Purchasing Power in 1996 Dollars
1955	0.75	4.39
1960	1.00	5.30
1965	1.25	6.23
1970	1.60	6.47
1975	2.10	6.12
1980	3.10	5.90
1985	3.35	4.88
1990	3.80	4.56
1995	4.25	4.38
2000	5.15	4.69
2005	5.15	4.15

Source: www.infoplease.com

EXAMPLE 1 Tracking the Minimum Wage

The numbers in Table 1.1 show the growth of the minimum hourly wage (MHW) from 1955 to 2005. The table also shows the MHW adjusted to the purchasing power of 1996 dollars (using the CPI-U, the Consumer Price Index for all Urban Consumers). Answer the following questions using only the data in the table.

(a) In what five-year period did the actual MHW increase the most?

(b) In what year did a worker earning the MHW enjoy the greatest purchasing power?

(c) A worker on minimum wage in 1980 was earning nearly twice as much as a worker on minimum wage in 1970, and yet there was great pressure to raise the minimum wage again. Why?

SOLUTION

(a) In the period 1975 to 1980 it increased by $1.00. Notice that the minimum wage never goes down, so we can tell that there were no other increases of this magnitude even though the table does not give data from every year.

(b) In 1970.

(c) Although the MHW increased from $1.60 to $3.10 in that period, the purchasing power actually dropped by $0.57 (in 1996 dollars). This is one way inflation can affect the economy.

Now try Exercise 11.

The numbers in Table 1.1 provide a numerical model for one aspect of the U.S. economy by using another numerical model, the urban Consumer Price Index (CPI-U), to adjust the data. Working with large numerical models is standard operating procedure in business and industry, where computers are relied upon to provide fast and accurate data processing.

EXAMPLE 2 Analyzing Prison Populations

Table 1.2 shows the growth in the number of prisoners incarcerated in state and federal prisons at year's end for selected years between 1980 and 2005. Is the proportion of female prisoners increasing over the years?

Table 1.2 U.S. Prison Population (thousands)

Year	Total	Male	Female
1980	329	316	13
1985	502	479	23
1990	774	730	44
1995	1125	1057	68
2000	1391	1298	93
2005	1526	1418	108

Source: U.S. Justice Department.

SOLUTION The *number* of female prisoners over the years is certainly increasing, but so is the total number of prisoners, so it is difficult to discern from the data whether the *proportion* of female prisoners is increasing. What we need is another column of numbers showing the ratio of female prisoners to total prisoners.

We could compute all the ratios separately, but it is easier to do this kind of repetitive calculation with a single command on a computer spreadsheet. You can also do this on a graphing calculator by manipulating lists (see Exercise 19). Table 1.3 shows the percentage of the total population each year that consists of female prisoners. With these data to extend our numerical model, it is clear that the proportion of female prisoners is increasing. ***Now try Exercise 19.***

Table 1.3 Female Percentage of U.S. Prison Population

Year	Female
1980	3.9
1985	4.6
1990	5.7
1995	6
2000	6.7
2005	7.1

Source: U.S. Justice Department.

Algebraic Models

An **algebraic model** uses formulas to relate variable quantities associated with the phenomena being studied. The added power of an algebraic model over a numerical model is that it can be used to generate numerical values of unknown quantities by relating them to known quantities.

EXAMPLE 3 Comparing Pizzas

A pizzeria sells a rectangular 18″ by 24″ pizza for the same price as its large round pizza (24″ diameter). If both pizzas are of the same thickness, which option gives the most pizza for the money?

SOLUTION We need to compare the *areas* of the pizzas. Fortunately, geometry has provided algebraic models that allow us to compute the areas from the given information.

For the rectangular pizza:

$$Area = l \times w = 18 \times 24 = 432 \text{ square inches.}$$

For the circular pizza:

$$Area = \pi r^2 = \pi\left(\frac{24}{2}\right)^2 = 144\pi \approx 452.4 \text{ square inches.}$$

The round pizza is larger and therefore gives more for the money.

Now try Exercise 21.

The algebraic models in Example 3 come from geometry, but you have probably encountered algebraic models from many other sources in your algebra and science courses.

Exploration Extensions

Suppose that after the sale, the merchandise prices are increased by 25%. If m represents the marked price before the sale, find an algebraic model for the post-sale price, including tax.

EXPLORATION 1 Designing an Algebraic Model

A department store is having a sale in which everything is discounted 25% off the marked price. The discount is taken at the sales counter, and then a state sales tax of 6.5% and a local sales tax of 0.5% are added on.

1. The discount price d is related to the marked price m by the formula $d = km$, where k is a certain constant. What is k?
2. The actual sale price s is related to the discount price d by the formula $s = d + td$, where t is a constant related to the total sales tax. What is t?
3. Using the answers from steps 1 and 2 you can find a constant p that relates s directly to m by the formula $s = pm$. What is p?
4. If you only have \$30, can you afford to buy a shirt marked \$36.99?
5. If you have a credit card but are determined to spend no more than \$100, what is the maximum total value of your marked purchases before you present them at the sales counter?

The ability to generate numbers from formulas makes an algebraic model far more useful as a predictor of behavior than a numerical model. Indeed, one optimistic goal of scientists and mathematicians when modeling phenomena is to fit an algebraic model to numerical data and then (even more optimistically) to analyze why it works. Not all models can be used to make accurate predictions. For example, nobody has ever devised a successful formula for predicting the ups and downs of the stock market as a function of time, although that does not stop investors from trying.

If numerical data do behave reasonably enough to suggest that an algebraic model might be found, it is often helpful to look at a picture first. That brings us to graphical models.

Graphical Models

A **graphical model** is a visible representation of a numerical model or an algebraic model that gives insight into the relationships between variable quantities. Learning to interpret and use graphs is a major goal of this book.

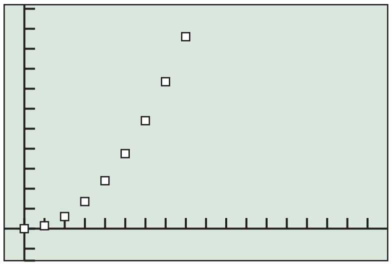
EXAMPLE 4 Visualizing Galileo's Gravity Experiments

Galileo Galilei (1564–1642) spent a good deal of time rolling balls down inclined planes, carefully recording the distance they traveled as a function of elapsed time. His experiments are commonly repeated in physics classes today, so it is easy to reproduce a typical table of Galilean data.

Elapsed time (seconds)	0	1	2	3	4	5	6	7	8
Distance traveled (inches)	0	0.75	3	6.75	12	18.75	27	36.75	48

What graphical model fits the data? Can you find an algebraic model that fits?

(continued)



[–1, 18] by [–8, 56]

FIGURE 1.1 A scatter plot of the data from a Galileo gravity experiment. (Example 4)

SOLUTION A scatter plot of the data is shown in Figure 1.1.

Galileo's experience with quadratic functions suggested to him that this figure was a parabola with its vertex at the origin; he therefore modeled the effect of gravity as a quadratic function:

$$d = kt^2.$$

Because the ordered pair $(1, 0.75)$ must satisfy the equation, it follows that $k = 0.75$, yielding the equation

$$d = 0.75t^2.$$

You can verify numerically that this algebraic model correctly predicts the rest of the data points. We will have much more to say about parabolas in Chapter 2.

Now try Exercise 23.

This insight led Galileo to discover several basic laws of motion that would eventually be named after Isaac Newton. While Galileo had found the algebraic model to describe the path of the ball, it would take Newton's calculus to explain why it worked.

EXAMPLE 5 Fitting a Curve to Data

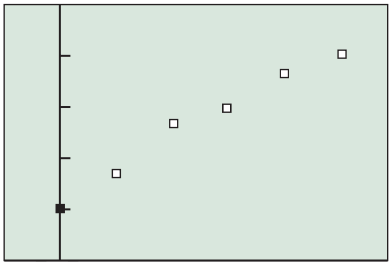
We showed in Example 2 that the percentage of females in the U.S. prison population has been steadily growing over the years. Model this growth graphically and use the graphical model to suggest an algebraic model.

SOLUTION Let t be the number of years after 1980, and let F be the percentage of females in the prison population from year 0 to year 25. From the data in Table 1.3 we get the corresponding data in Table 1.4:

Table 1.4 Percentage (F) of Females in the Prison Population t years after 1980

t	0	5	10	15	20	25
F	3.9	4.6	5.7	6.0	6.7	7.1

Source: U.S. Justice Department.



[–2, 28] by [3, 8]

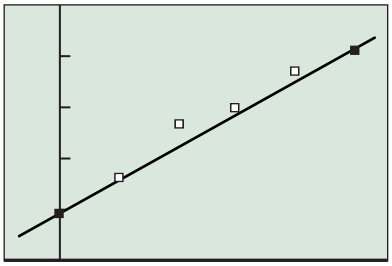
FIGURE 1.2 A scatter plot of the data in Table 1.4. (Example 5)

A scatter plot of the data is shown in Figure 1.2.

This pattern looks linear. If we use a line as our graphical model, we can find an algebraic model by finding the equation of the line. We will describe in Chapter 2 how a statistician would find the best line to fit the data, but we can get a pretty good fit for now by finding the line through the points $(0, 3.9)$ and $(25, 7.1)$.

The slope is $(7.1 - 3.9)/(25 - 0) = 0.128$ and the y-intercept is 3.9. Therefore, the line has equation $y = 0.128x + 3.9$. You can see from Figure 1.3 that this line does a very nice job of modeling the data.

Now try Exercises 13 and 15.



[–2, 28] by [3, 8]

FIGURE 1.3 The line with equation $y = 0.128x + 3.9$ is a good model for the data in Table 1.4. (Example 5)

Exploration Extensions

What are the advantages of a linear model over a quadratic model for these data?

EXPLORATION 2 Interpreting the Model

The parabola in Example 4 arose from a law of physics that governs falling objects, which should inspire more confidence than the linear model in Example 5. We can repeat Galileo's experiment many times with differently sloped ramps, with different units of measurement, and even on different planets, and a quadratic model will fit it every time. The purpose of this Exploration is to think more deeply about the linear model in the prison example.

1. The linear model we found will not continue to predict the percentage of female prisoners in the United States indefinitely. Why must it eventually fail?
2. Do you think that our linear model will give an accurate estimate of the percentage of female prisoners in the United States in 2009? Why or why not?
3. The linear model is such a good fit that it actually calls our attention to the unusual jump in the percentage of female prisoners in 1990. Statisticians would look for some unusual "confounding" factor in 1990 that might explain the jump. What sort of factors do you think might explain it?
4. Does Table 1.1 suggest a possible factor that might influence female crime statistics?

Prerequisite Chapter

In the Prerequisite chapter we defined solution of an equation, solving an equation, x-intercept, and graph of an equation in x and y.

There are other ways of graphing numerical data that are particularly useful for statistical studies. We will treat some of them in Chapter 9. The scatter plot will be our choice of data graph for the time being, as it provides the closest connection to graphs of functions in the Cartesian plane.

The Zero Factor Property

The main reason for studying algebra through the ages has been to solve equations. We develop algebraic models for phenomena so that we can solve problems, and the solutions to the problems usually come down to finding solutions of algebraic equations.

If we are fortunate enough to be solving an equation in a single variable, we might proceed as in the following example.

EXAMPLE 6 Solving an Equation Algebraically

Find all real numbers x for which $6x^3 = 11x^2 + 10x$.

SOLUTION We begin by changing the form of the equation to $6x^3 - 11x^2 - 10x = 0$.

We can then solve this equation algebraically by factoring:

$$6x^3 - 11x^2 - 10x = 0$$
$$x(6x^2 - 11x - 10) = 0$$
$$x(2x - 5)(3x + 2) = 0$$
$$x = 0 \quad \text{or} \quad 2x - 5 = 0 \quad \text{or} \quad 3x + 2 = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{5}{2} \quad \text{or} \quad x = -\frac{2}{3}$$

Now try Exercise 31.

In Example 6, we used the important Zero Factor Property of real numbers.

The Zero Factor Property

A product of real numbers is zero if and only if at least one of the factors in the product is zero.

It is this property that algebra students use to solve equations in which an expression is set equal to zero. Modern problem solvers are fortunate to have an alternative way to find such solutions.

If we graph the expression, then the x-intercepts of the graph of the expression will be the values for which the expression equals 0.

EXAMPLE 7 Solving an Equation: Comparing Methods

Solve the equation $x^2 = 10 - 4x$.

SOLUTION

Solve Algebraically

The given equation is equivalent to $x^2 + 4x - 10 = 0$.

This quadratic equation has irrational solutions that can be found by the quadratic formula.

$$x = \frac{-4 + \sqrt{16 + 40}}{2} \approx 1.7416574$$

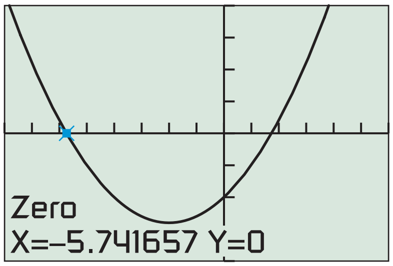
and

$$x = \frac{-4 - \sqrt{16 + 40}}{2} \approx -5.7416574$$

While the decimal answers are certainly accurate enough for all practical purposes, it is important to note that only the expressions found by the quadratic formula give the *exact* real number answers. The tidiness of exact answers is a worthy mathematical goal. Realistically, however, exact answers are often impossible to obtain, even with the most sophisticated mathematical tools.

Solve Graphically

We first find an equivalent equation with 0 on the right-hand side: $x^2 + 4x - 10 = 0$. We next graph the equation $y = x^2 + 4x - 10$, as shown in Figure 1.4.



[–8, 6] by [–20, 20]

FIGURE 1.4 The graph of $y = x^2 + 4x - 10$. (Example 7)

We then use the grapher to locate the x-intercepts of the graph:

$$x \approx 1.7416574 \text{ and } x \approx -5.741657.$$

Now try Exercise 35.

Solving Equations with Technology

Example 7 shows one method of solving an equation with technology. Some graphers could also solve the equation in Example 7 by finding the *intersection* of the graphs of $y = x^2$ and $y = 10 - 4x$. Some graphers have built-in equation solvers. Each method has its advantages and disadvantages, but we recommend the "finding the x-intercepts" technique for now because it most closely parallels the classical algebraic techniques for finding roots of equations, and makes the connection between the algebraic and graphical models easier to follow and appreciate.

We used the graphing utility of the calculator to **solve graphically** in Example 7. Most calculators also have solvers that would enable us to **solve numerically** for the same decimal approximations without considering the graph. Some calculators have computer algebra systems that will solve numerically to produce exact answers in certain cases. In this book we will distinguish between these two technological methods and the traditional pencil-and-paper methods used to **solve algebraically**.

Every method of solving an equation usually comes down to finding where an expression equals zero. If we use $f(x)$ to denote an algebraic expression in the variable x, the connections are as follows:

Fundamental Connection

If a is a real number that solves the equation $f(x) = 0$, then these three statements are equivalent:

1. The number a is a **root** (or **solution**) of the **equation** $f(x) = 0$.
2. The number a is a **zero** of $y = f(x)$.
3. The number a is an ***x*-intercept** of the **graph** of $y = f(x)$. (Sometimes the point $(a, 0)$ is referred to as an x-intercept.)

Problem Solving

George Pólya (1887–1985) is sometimes called the father of modern problem solving, not only because he was good at it (as he certainly was) but also because he published the most famous analysis of the problem-solving process: *How to Solve It: A New Aspect of Mathematical Method*. His "four steps" are well known to most mathematicians:

Pólya's Four Problem-Solving Steps

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look back.

The problem-solving process that we recommend you use throughout this course will be the following version of Pólya's four steps.

A Problem-Solving Process

Step 1—Understand the problem.

- Read the problem as stated, several times if necessary.
- Be sure you understand the meaning of each term used.
- Restate the problem in your own words. Discuss the problem with others if you can.
- Identify clearly the information that you need to solve the problem.
- Find the information you need from the given data.

Step 2—Develop a mathematical model of the problem.

- Draw a picture to visualize the problem situation. It usually helps.
- Introduce a variable to represent the quantity you seek. (In some cases there may be more than one.)
- Use the statement of the problem to find an equation or inequality that relates the variables you seek to quantities that you know.

Step 3—Solve the mathematical model and support or confirm the solution.

- **Solve algebraically** using traditional algebraic methods and **support graphically or support numerically** using a graphing utility.
- **Solve graphically or numerically** using a graphing utility and **confirm algebraically** using traditional algebraic methods.
- **Solve graphically or numerically** because there is no other way possible.

Step 4—Interpret the solution in the problem setting.

- Translate your mathematical result into the problem setting and decide whether the result makes sense.

EXAMPLE 8 Applying the Problem-Solving Process

The engineers at an auto manufacturer pay students \$0.08 per mile plus \$25 per day to road test their new vehicles.

(a) How much did the auto manufacturer pay Sally to drive 440 miles in one day?

(b) John earned \$93 test-driving a new car in one day. How far did he drive?

SOLUTION

Model

A picture of a car or of Sally or John would not be helpful, so we go directly to designing the model. Both John and Sally earned \$25 for one day, plus \$0.08 per mile. Multiply dollars/mile by miles to get dollars.

So if p represents the pay for driving x miles in one day, our algebraic model is

$$p = 25 + 0.08x.$$

Solve Algebraically

(a) To get Sally's pay we let $x = 440$ and solve for p:

$$\begin{aligned} p &= 25 + 0.08(440) \\ &= 60.20 \end{aligned}$$

(b) To get John's mileage we let $p = 93$ and solve for x:

$$\begin{aligned} 93 &= 25 + 0.08x \\ 68 &= 0.08x \\ x &= \frac{68}{0.08} \\ x &= 850 \end{aligned}$$

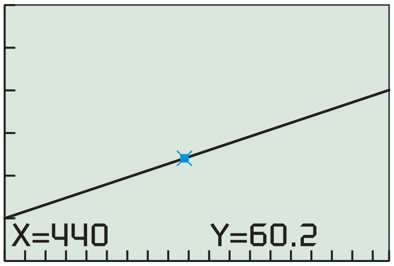
Support Graphically

Figure 1.5a shows that the point $(440, 60.20)$ is on the graph of $y = 25 + 0.08x$, supporting our answer to (a). Figure 1.5b shows that the point $(850, 93)$ is on the graph of $y = 25 + 0.08x$, supporting our answer to (b). (We could also have **supported** our answer **numerically** by simply substituting in for each x and confirming the value of p.)

Interpret

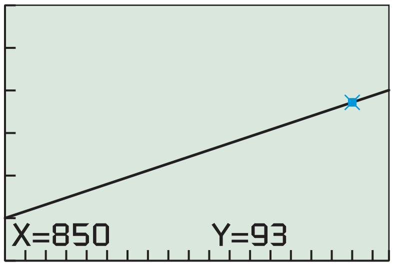
Sally earned \$60.20 for driving 440 miles in one day. John drove 850 miles in one day to earn \$93.00.

Now try Exercise 47.



[0, 940] by [0, 150]

(a)



[0, 940] by [0, 150]

(b)

FIGURE 1.5 Graphical support for the algebraic solutions in Example 8.

It is not really necessary to *show* written support as part of an algebraic solution, but it is good practice to support answers wherever possible simply to reduce the chance for error. We will often show written support of our solutions in this book in order to highlight the connections among the algebraic, graphical, and numerical models.

Grapher Failure and Hidden Behavior

While the graphs produced by computers and graphing calculators are wonderful tools for understanding algebraic models and their behavior, it is important to keep in mind that machines have limitations. Occasionally they can produce graphical models that misrepresent the phenomena we wish to study, a problem we call **grapher failure**. Sometimes the viewing window will be too large, obscuring details of the graph which we call **hidden behavior**. We will give an example of each just to illustrate what can happen, but rest assured that these difficulties rarely occur with graphical models that arise from real-world problems.

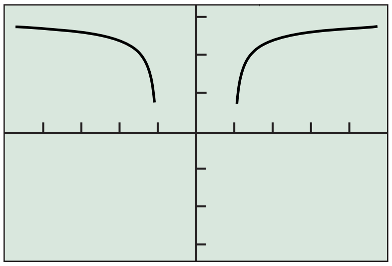
EXAMPLE 9 Seeing Grapher Failure

Look at the graph of $y = 3 - \dfrac{1}{\sqrt{x^2 - 1}}$ in the ZDecimal window on a graphing calculator. Are there any x-intercepts?

SOLUTION The graph is shown in Figure 1.6a.

Technology Note

One way to get the table in Figure 1.6b is to use the "Ask" feature of your graphing calculator and enter each x-value separately.



[–4.7, 4.7] by [–3.1, 3.1]

(a)

X	Y1
.8	ERROR
.9	ERROR
1	ERROR
1.1	.81782
1.2	1.4924
1.3	1.7961
1.4	1.9794

Y1 = 3–1/√(X²–1)

(b)

FIGURE 1.6 (a) A graph with no apparent intercepts. (b) The function $y = 3 - 1/\sqrt{x^2 - 1}$ is undefined when $|x| \le 1$.

The graph seems to have no x-intercepts, yet we can find some by solving the equation $0 = 3 - 1/\sqrt{x^2 - 1}$ algebraically:

$$\begin{aligned} 0 &= 3 - 1/\sqrt{x^2 - 1} \\ 1/\sqrt{x^2 - 1} &= 3 \\ \sqrt{x^2 - 1} &= 1/3 \\ x^2 - 1 &= 1/9 \\ x^2 &= 10/9 \\ x &= \pm\sqrt{10/9} \approx \pm 1.054 \end{aligned}$$

There should be x-intercepts at about ± 1.054. What went wrong?

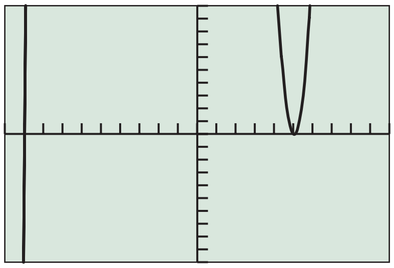
The answer is a simple form of grapher failure. As the table shows, the function is undefined for the sampled x-values until $x = 1.1$, at which point the graph "turns on," beginning with the pixel at $(1.1, 0.81782)$ and continuing from there to the right. Similarly, the graph coming from the left "turns off" at $x = -1$, before it gets to the x-axis. The x-intercepts might well appear in other windows, but for this particular function in this particular window, the behavior we expect to see is not there.

Now try Exercise 49.

EXAMPLE 10 Not Seeing Hidden Behavior

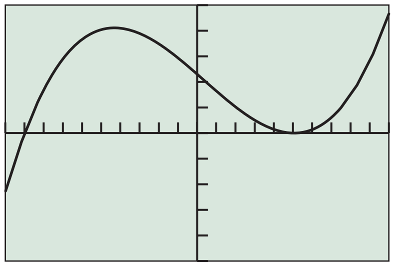
Solve graphically: $x^3 - 1.1x^2 - 65.4x + 229.5 = 0$.

SOLUTION Figure 1.7a shows the graph in the standard $[-10, 10]$ by $[-10, 10]$ window, an inadequate choice because too much of the graph is off the screen. Our horizontal dimensions look fine, so we adjust our vertical dimensions to $[-500, 500]$, yielding the graph in Figure 1.7b.



[–10, 10] by [–10, 10]

(a)



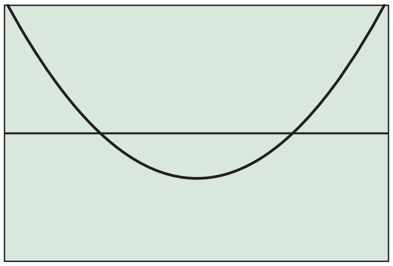
[–10, 10] by [–500, 500]

(b)

FIGURE 1.7 The graph of $y = x^3 - 1.1x^2 - 65.4x + 229.5$ in two viewing windows. (Example 10)

We use the grapher to locate an x-intercept near -9 (which we find to be -9) and then an x-intercept near 5 (which we find to be 5). The graph leads us to believe that we have finished. However, if we zoom in closer to observe the behavior near $x = 5$, the graph tells a new story (Figure 1.8).

In this graph we see that there are actually *two* x-intercepts near 5 (which we find to be 5 and 5.1). There are therefore three roots (or zeros) of the equation $x^3 - 1.1x^2 - 65.4x + 229.5 = 0$: $x = -9$, $x = 5$, and $x = 5.1$. ***Now try Exercise 51.***



[4.95, 5.15] by [–0.1, 0.1]

FIGURE 1.8 A closer look at the graph of $y = x^3 - 1.1x^2 - 65.4x + 229.5$. (Example 10)

You might wonder if there could be still *more* hidden x-intercepts in Example 10! We will learn in Chapter 2 how the *Fundamental Theorem of Algebra* guarantees that there are not.

A Word About Proof

While Example 10 is still fresh in our minds, let us point out a subtle, but very important, consideration about our solution.

We *solved graphically* to find two solutions, then eventually three solutions, to the given equation. Although we did not show the steps, it is easy to *confirm numerically* that the three numbers found are actually solutions by substituting them into the equation. But the problem asked us to find *all* solutions. While we could explore that equation graphically in a hundred more viewing windows and never find another solution, our failure to find them would not *prove* that they are not out there somewhere. That is why the Fundamental Theorem of Algebra is so important. It tells us that there can be at most three real solutions to *any* cubic equation, so we know for a fact that there are no more.

Exploration is encouraged throughout this book because it is how mathematical progress is made. Mathematicians are never satisfied, however, until they have *proved* their results. We will show you proofs in later chapters and we will ask you to produce proofs occasionally in the exercises. That will be a time for you to set the technology aside, get out a pencil, and show in a logical sequence of algebraic steps that something is undeniably and universally true. This process is called **deductive reasoning**.

Teacher Note

Sometimes it is impossible to show all of the details of a graph in a single window. For example, in Example 10 the graph in Figure 1.8 reveals minute details of the graph, but it hides the overall shape of the graph.

EXAMPLE 11 Proving a Peculiar Number Fact

Prove that 6 is a factor of $n^3 - n$ for every positive integer n.

SOLUTION You can explore this expression for various values of n on your calculator. Table 1.5 shows it for the first 12 values of n.

Table 1.5 The First 12 Values of $n^3 - n$

n	1	2	3	4	5	6	7	8	9	10	11	12
$n^3 - n$	0	6	24	60	120	210	336	504	720	990	1320	1716

All of these numbers are divisible by 6, but that does not prove that they will continue to be divisible by 6 for all values of n. In fact, a table with a billion values, all divisible by 6, would not constitute a proof. Here is a proof:

Let n be *any* positive integer.

- We can factor $n^3 - n$ as the product of three numbers: $(n - 1)(n)(n + 1)$.
- The factorization shows that $n^3 - n$ is always the product of three consecutive integers.
- Every set of three consecutive integers must contain a multiple of 3.
- Since 3 divides a factor of $n^3 - n$, it follows that 3 is a factor of $n^3 - n$ itself.
- Every set of three consecutive integers must contain a multiple of 2.
- Since 2 divides a factor of $n^3 - n$, it follows that 2 is a factor of $n^3 - n$ itself.
- Since both 2 and 3 are factors of $n^3 - n$, we know that 6 is a factor of $n^3 - n$.

End of proof! *Now try Exercise 53.*

QUICK REVIEW 1.1 *(For help, go to Section A.2.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

Factor the following expressions completely over the real numbers.

1. $x^2 - 16$
2. $x^2 + 10x + 25$
3. $81y^2 - 4$
4. $3x^3 - 15x^2 + 18x$
5. $16h^4 - 81$
6. $x^2 + 2xh + h^2$
7. $x^2 + 3x - 4$
8. $x^2 - 3x + 4$
9. $2x^2 - 11x + 5$
10. $x^4 + x^2 - 20$

SECTION 1.1 EXERCISES

In Exercises 1–10, match the numerical model to the corresponding graphical model (a–j) and algebraic model (k–t).

1.

x	3	5	7	9	12	15
y	6	10	14	18	24	30

2.

x	0	1	2	3	4	5
y	2	3	6	11	18	27

3.

x	2	4	6	8	10	12
y	4	10	16	22	28	34

4.

x	5	10	15	20	25	30
y	90	80	70	60	50	40

5.

x	1	2	3	4	5	6
y	39	36	31	24	15	4

6.

x	1	2	3	4	5	6
y	5	7	9	11	13	15

7.

x	5	7	9	11	13	15
y	1	2	3	4	5	6

8.

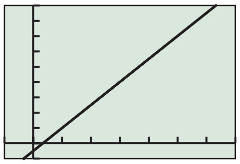
x	4	8	12	14	18	24
y	20	72	156	210	342	600

9.

x	3	4	5	6	7	8
y	8	15	24	35	48	63

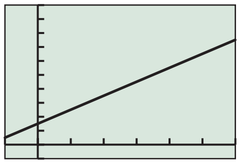
10.

x	4	7	12	19	28	39
y	1	2	3	4	5	6



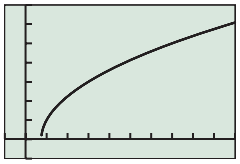
[–2, 14] by [–4, 36]

(a)



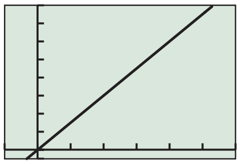
[–1, 6] by [–2, 20]

(b)



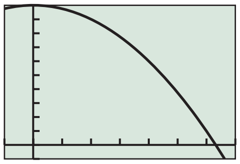
[–4, 40] by [–1, 7]

(c)



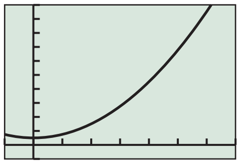
[–3, 18] by [–2, 32]

(d)



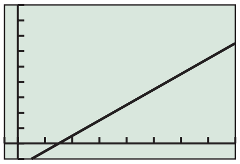
[–1, 7] by [–4, 40]

(e)



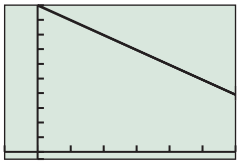
[–1, 7] by [–4, 40]

(f)



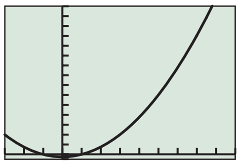
[–1, 16] by [–1, 9]

(g)



[–5, 30] by [–5, 100]

(h)



[–3, 9] by [–2, 60]

(i)

[–5, 40] by [–10, 650]

(j)

(k) $y = x^2 + x$ **(l)** $y = 40 - x^2$

(m) $y = (x + 1)(x - 1)$ **(n)** $y = \sqrt{x - 3}$

(o) $y = 100 - 2x$ **(p)** $y = 3x - 2$

(q) $y = 2x$ **(r)** $y = x^2 + 2$

(s) $y = 2x + 3$ **(t)** $y = \dfrac{x - 3}{2}$

Exercises 11–18 refer to the data in Table 1.6 below, showing the percentage of the female and male populations in the United States employed in the civilian work force in selected years from 1954 to 2004.

Table 1.6 Employment Statistics

Year	Female	Male
1954	32.3	83.5
1959	35.1	82.3
1964	36.9	80.9
1969	41.1	81.1
1974	42.8	77.9
1979	47.7	76.5
1984	50.1	73.2
1989	54.9	74.5
1994	56.2	72.6
1999	58.5	74.0
2004	57.4	71.9

Source: www.bls.gov

11. (a) According to the numerical model, what has been the trend in females joining the work force since 1954?

(b) In what 5-year interval did the percentage of women who were employed change the most?

12. (a) According to the numerical model, what has been the trend in males joining the work force since 1954?

(b) In what 5-year interval did the percentage of men who were employed change the most?

13. Model the data graphically with two scatter plots on the same graph, one showing the percentage of women employed as a function of time and the other showing the same for men. Measure time in years since 1954.

14. Are the male percentages falling faster than the female percentages are rising, or vice versa?

15. Model the data algebraically with linear equations of the form $y = mx + b$. Write one equation for the women's data and another equation for the men's data. Use the 1954 and 1999 ordered pairs to compute the slopes.

16. If the percentages continue to follow the linear models you found in Exercise 15, what will the employment percentages for women and men be in the year 2009?

17. If the percentages continue to follow the linear models you found in Exercise 15, when will the percentages of women and men in the civilian work force be the same? What percentage will that be?

18. Writing to Learn Explain why the percentages cannot continue indefinitely to follow the linear models that you wrote in Exercise 15.

19. Doing Arithmetic with Lists Enter the data from the "Total" column of Table 1.2 of Example 2 into list L_1 in your calculator. Enter the data from the "Female" column into list L_2. Check a few computations to see that the procedures in (a) and (b) cause the calculator to divide each element of L_2 by the corresponding entry in L_1, multiply it by 100, and store the resulting list of percentages in L_3.

(a) On the home screen, enter the command: $100 \times L_2/L_1 \rightarrow L_3$.

(b) Go to the top of list L_3 and enter $L_3 = 100(L_2/L_1)$.

20. Comparing Cakes A bakery sells a 9″ by 13″ cake for the same price as an 8″ diameter round cake. If the round cake is twice the height of the rectangular cake, which option gives the most cake for the money?

21. Stepping Stones A garden shop sells 12″ by 12″ square stepping stones for the same price as 13″ round stones. If all of the stepping stones are the same thickness, which option gives the most rock for the money?

22. Free Fall of a Smoke Bomb At the Oshkosh, WI, air show, Jake Trouper drops a smoke bomb to signal the official beginning of the show. Ignoring air resistance, an object in free fall will fall d feet in t seconds, where d and t are related by the algebraic model $d = 16t^2$.

(a) How long will it take the bomb to fall 180 feet?

(b) If the smoke bomb is in free fall for 12.5 seconds after it is dropped, how high was the airplane when the smoke bomb was dropped?

23. Physics Equipment A physics student obtains the following data involving a ball rolling down an inclined plane, where t is the elapsed time in seconds and y is the distance traveled in inches.

t	0	1	2	3	4	5
y	0	1.2	4.8	10.8	19.2	30

Find an algebraic model that fits the data.

24. U.S. Air Travel The number of revenue passengers enplaned in the United States over the 14-year period from 1994 to 2007 is shown in Table 1.7.

Table 1.7 U.S. Air Travel

Year	Passengers (millions)	Year	Passengers (millions)
1994	528.8	2001	622.1
1995	547.8	2002	614.1
1996	581.2	2003	646.5
1997	594.7	2004	702.9
1998	612.9	2005	738.3
1999	636.0	2006	744.2
2000	666.2	2007	769.2

Source: www.airlines.org

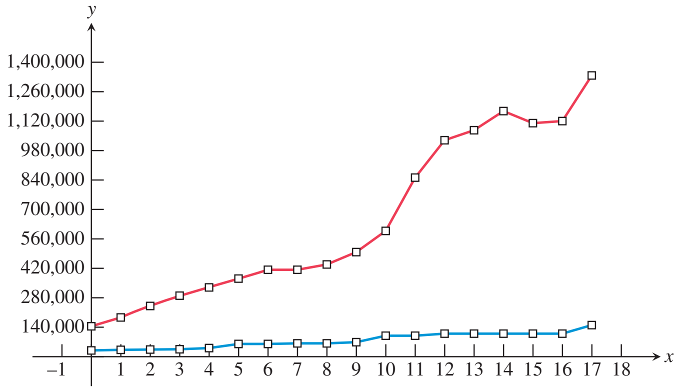
(a) Graph a scatter plot of the data. Let x be the number of years since 1994.

(b) From 1994 to 2000 the data seem to follow a linear model. Use the 1994 and 2000 points to find an equation of the line and superimpose the line on the scatter plot.

(c) According to the linear model, in what year did the number of passengers seem destined to reach 900 million?

(d) What happened to disrupt the linear model?

Exercises 25–28 refer to the graph below, which shows the *minimum* salaries in major league baseball over a recent 18-year period and the *average* salaries in major league baseball over the same period. Salaries are measured in dollars and time is measured after the starting year (year 0).



Source: Major League Baseball Players Association.

25. Which line is which, and how do you know?

26. After Peter Ueberroth's resignation as baseball commissioner in 1988 and his successor's untimely death in 1989, the team owners broke free of previous restrictions and began an era of competitive spending on player salaries. Identify where the 1990 salaries appear in the graph and explain how you can spot them.

27. The owners attempted to halt the uncontrolled spending by proposing a salary cap, which prompted a players' strike in 1994. The strike caused the 1995 season to be shortened and left many fans angry. Identify where the 1995 salaries appear in the graph and explain how you can spot them.

28. Writing to Learn Analyze the general patterns in the graphical model and give your thoughts about what the long-term implications might be for

(a) the players;

(b) the team owners;

(c) the baseball fans.

In Exercises 29–38, solve the equation algebraically and confirm graphically.

29. $v^2 - 5 = 8 - 2v^2$

30. $(x + 11)^2 = 121$

31. $2x^2 - 5x + 2 = (x - 3)(x - 2) + 3x$

32. $x^2 - 7x - \frac{3}{4} = 0$

33. $x(2x - 5) = 12$

34. $x(2x - 1) = 10$

35. $x(x + 7) = 14$

36. $x^2 - 3x + 4 = 2x^2 - 7x - 8$

37. $x + 1 - 2\sqrt{x + 4} = 0$

38. $\sqrt{x} + x = 1$

In Exercises 39–46, solve the equation graphically by converting it to an equivalent equation with 0 on the right-hand side and then finding the x-intercepts.

39. $2x - 5 = \sqrt{x + 4}$

40. $|3x - 2| = 2\sqrt{x + 8}$

41. $|2x - 5| = 4 - |x - 3|$

42. $\sqrt{x + 6} = 6 - 2\sqrt{5 - x}$

43. $2x - 3 = x^3 - 5$

44. $x + 1 = x^3 - 2x - 5$

45. $(x + 1)^{-1} = x^{-1} + x$

46. $x^2 = |x|$

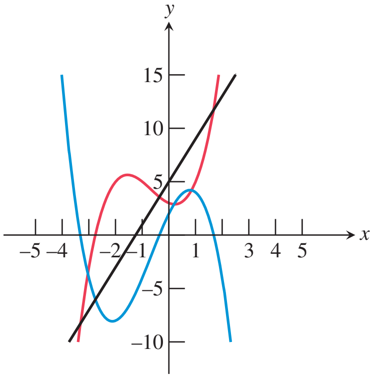
47. Swan Auto Rental charges \$32 per day plus \$0.18 per mile for an automobile rental.

(a) Elaine rented a car for one day and she drove 83 miles. How much did she pay?

(b) Ramon paid \$69.80 to rent a car for one day. How far did he drive?

48. Connecting Graphs and Equations The curves on the graph below are the graphs of the three curves given by

$y_1 = 4x + 5$
$y_2 = x^3 + 2x^2 - x + 3$
$y_3 = -x^3 - 2x^2 + 5x + 2.$



(a) Write an equation that can be solved to find the points of intersection of the graphs of y_1 and y_2.

(b) Write an equation that can be solved to find the x-intercepts of the graph of y_3.

(c) Writing to Learn How does the graphical model reflect the fact that the answers to (a) and (b) are equivalent algebraically?

(d) Confirm numerically that the x-intercepts of y_3 give the same values when substituted into the expressions for y_1 and y_2.

49. Exploring Grapher Failure Let $y = (x^{200})^{1/200}$.

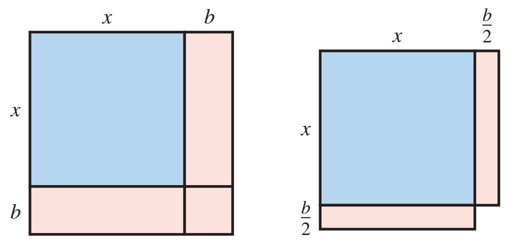
(a) Explain algebraically why $y = x$ for all $x \geq 0$.

(b) Graph the equation $y = (x^{200})^{1/200}$ in the window [0, 1] by [0, 1].

(c) Is the graph different from the graph of $y = x$?

(d) Can you explain why the grapher failed?

50. Connecting Algebra and Geometry Explain how the algebraic equation $(x + b)^2 = x^2 + 2bx + b^2$ models the areas of the regions in the geometric figure shown below on the left:



(Ex. 50) (Ex. 52)

51. Exploring Hidden Behavior Solving graphically, find all real solutions to the following equations. Watch out for hidden behavior.

(a) $y = 10x^3 + 7.5x^2 - 54.85x + 37.95$

(b) $y = x^3 + x^2 - 4.99x + 3.03$

52. Connecting Algebra and Geometry The geometric figure shown on the right above is a large square with a small square missing.

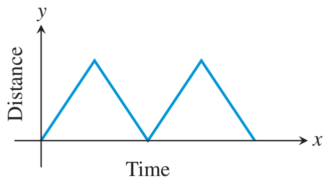
(a) Find the area of the figure.

(b) What area must be added to complete the large square?

(c) Explain how the algebraic formula for completing the square models the completing of the square in (b).

53. Proving a Theorem Prove that if n is a positive integer, then $n^2 + 2n$ is either odd or a multiple of 4. Compare your proof with those of your classmates.

54. Writing to Learn The graph below shows the distance from home against time for a jogger. Using information from the graph, write a paragraph describing the jogger's workout.



Standardized Test Questions

55. True or False A product of real numbers is zero if and only if every factor in the product is zero. Justify your answer.

56. True or False An algebraic model can always be used to make accurate predictions.

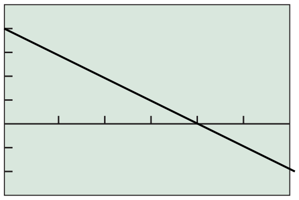
In Exercises 57–60, you may use a graphing calculator to decide which algebraic model corresponds to the given graphical or numerical model.

(A) $y = 2x + 3$ **(B)** $y = x^2 + 5$

(C) $y = 12 - 3x$ **(D)** $y = 4x + 3$

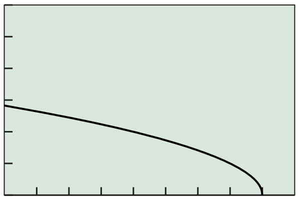
(E) $y = \sqrt{8 - x}$

57. Multiple Choice



[0, 6] by [–9, 15]

58. Multiple Choice



[0, 9] by [0, 6]

59. Multiple Choice

x	1	2	3	4	5	6
y	6	9	14	21	30	41

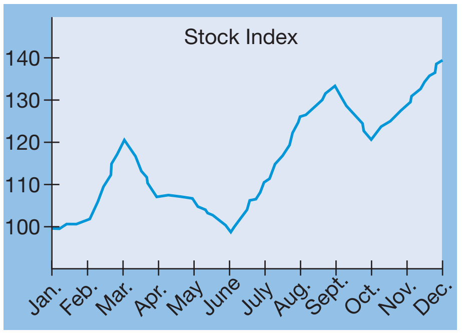
60. Multiple Choice

x	0	2	4	6	8	10
y	3	7	11	15	19	23

Explorations

61. Analyzing the Market Both Ahmad and LaToya watch the stock market throughout the year for stocks that make significant jumps from one month to another. When they spot one, each buys 100 shares. Ahmad's rule is to sell the stock if it fails to perform well for three months in a row. LaToya's rule is to sell in December if the stock has failed to perform well since its purchase.

The graph below shows the monthly performance in dollars (Jan–Dec) of a stock that both Ahmad and LaToya have been watching.



(a) Both Ahmad and LaToya bought the stock early in the year. In which month?

(b) At approximately what price did they buy the stock?

(c) When did Ahmad sell the stock?

(d) How much did Ahmad lose on the stock?

(e) Writing to Learn Explain why LaToya's strategy was better than Ahmad's for this particular stock in this particular year.

(f) Sketch a 12-month graph of a stock's performance that would favor Ahmad's strategy over LaToya's.

62. Group Activity Creating Hidden Behavior You can create your own graphs with hidden behavior. Working in groups of two or three, try this exploration.

(a) Graph the equation $y = (x + 2)(x^2 - 4x + 4)$ in the window $[-4, 4]$ by $[-10, 10]$.

(b) Confirm algebraically that this function has zeros only at $x = -2$ and $x = 2$.

(c) Graph the equation $y = (x + 2)(x^2 - 4x + 4.01)$ in the window $[-4, 4]$ by $[-10, 10]$

(d) Confirm algebraically that this function has only one zero, at $x = -2$. (Use the discriminant.)

(e) **Graph the equation** $(x + 2)(x^2 - 4x + 3.99)$ in the window $[-4, 4]$ by $[-10, 10]$.

(f) Confirm algebraically that this function has three zeros. (Use the discriminant.)

Extending the Ideas

63. **The Proliferation of Cell Phones** Table 1.8 shows the number of cellular phone subscribers in the United States and their average monthly bill in the years from 1998 to 2007.

Table 1.8 Cellular Phone Subscribers

Year	Subscribers (millions)	Average Local Monthly Bill ($)
1998	69.2	39.43
1999	86.0	41.24
2000	109.5	45.27
2001	128.4	47.37
2002	140.8	48.40
2003	158.7	49.91
2004	182.1	50.64
2005	207.9	49.98
2006	233.0	50.56
2007	255.4	49.79

Source: Cellular Telecommunication & Internet Association.

(a) Graph the scatter plots of the number of subscribers and the average local monthly bill as functions of time, letting time $t =$ the number of years after 1990.

(b) One of the scatter plots clearly suggests a linear model in the form $y = mx + b$. Use the points at $t = 8$ and $t = 16$ to find a linear model.

(c) Superimpose the graph of the linear model onto the scatter plot. Does the fit appear to be good?

(d) Why does a linear model seem inappropriate for the other scatter plot? Can you think of a function that might fit the data better?

(e) In 1995 there were 33.8 million subscribers with an average local monthly bill of $51.00. Add these points to the scatter plots.

(f) **Writing to Learn** The 1995 points do not seem to fit the models used to represent the 1998–2004 data. Give a possible explanation for this.

64. **Group Activity** (Continuation of Exercise 63) Discuss the economic forces suggested by the two models in Exercise 63 and speculate about the future by analyzing the graphs.