1.2 Functions and Their Properties

What you'll learn about

- Function Definition and Notation
- Domain and Range
- Continuity
- Increasing and Decreasing Functions
- Boundedness
- Local and Absolute Extrema
- Symmetry
- Asymptotes
- End Behavior

… and why

Functions and graphs form the basis for understanding the mathematics and applications you will see both in your workplace and in coursework in college.

In this section we will introduce the terminology that is used to describe functions throughout this book. Feel free to skim over parts with which you are already familiar, but take the time to become comfortable with concepts that might be new to you (like continuity and symmetry). Even if it takes several days to cover this section, it will be precalculus time well spent.

Function Definition and Notation

Mathematics and its applications abound with examples of formulas by which quantitative variables are related to each other. The language and notation of functions is ideal for that purpose. A function is actually a simple concept; if it were not, history would have replaced it with a simpler one by now. Here is the definition.

DEFINITION Function, Domain, and Range

A function from a set D to a set R is a rule that assigns to every element in D a unique element in R. The set D of all input values is the **domain** of the function, and the set R of all output values is the **range** of the function.

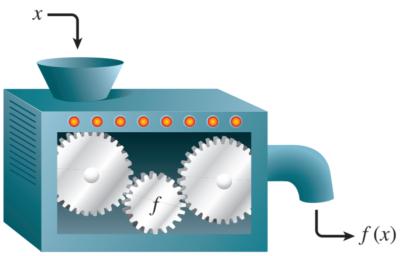


FIGURE 1.9 A "machine" diagram for a function.

There are many ways to look at functions. One of the most intuitively helpful is the "machine" concept (Figure 1.9), in which values of the domain (x) are fed into the machine (the function f) to produce range values (y). To indicate that y comes from the function acting on x, we use Euler's elegant **function notation** $y = f(x)$ (which we read as "**y equals f of x**" or "**the value of f at x**"). Here x is the **independent variable** and y is the **dependent variable**.

A function can also be viewed as a **mapping** of the elements of the domain onto the elements of the range. Figure 1.10a shows a function that maps elements from the domain X onto elements of the range Y. Figure 1.10b shows another such mapping, but *this one is not a function*, since the rule does not assign the element x_1 to a *unique* element of Y.

A Bit of History

The word *function* in its mathematical sense is generally attributed to Gottfried Leibniz (1646–1716), one of the pioneers in the methods of calculus. His attention to clarity of notation is one of his greatest contributions to scientific progress, which is why we still use his notation in calculus courses today. Ironically, it was not Leibniz but Leonhard Euler (1707–1783) who introduced the familiar notation $f(x)$.

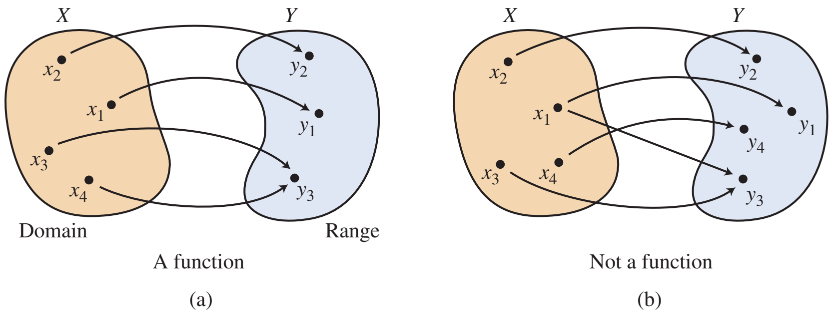


FIGURE 1.10 The diagram in (a) depicts a mapping from X to Y that is a function. The diagram in (b) depicts a mapping from X to Y that is not a function.

This uniqueness of the range value is very important to us as we study function behavior. Knowing that $f(2) = 8$ tells us something about f, and that understanding would be contradicted if we were to discover later that $f(2) = 4$. That is why you will never see a function defined by an ambiguous formula like $f(x) = 3x \pm 2$.

EXAMPLE 1 Defining a Function

Does the formula $y = x^2$ define y as a function of x?

SOLUTION Yes, y is a function of x. In fact, we can write the formula in function notation: $f(x) = x^2$. When a number x is substituted into the function, the square of x will be the output, and there is no ambiguity about what the square of x is.

Now try Exercise 3.

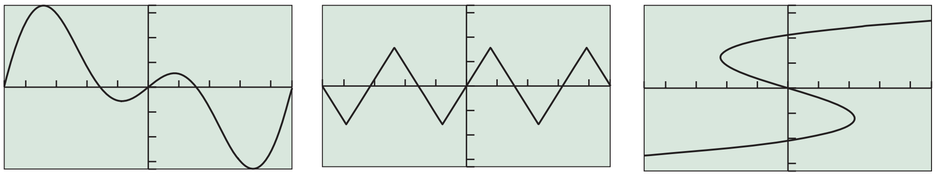
Another useful way to look at functions is graphically. The **graph of the function** $\boldsymbol{y = f(x)}$ is the set of all points $(x, f(x))$, x in the domain of f. We match domain values along the x-axis with their range values along the y-axis to get the ordered pairs that yield the graph of $y = f(x)$.

EXAMPLE 2 Seeing a Function Graphically

Of the three graphs shown in Figure 1.11, which is *not* the graph of a function? How can you tell?

SOLUTION The graph in (c) is not the graph of a function. There are three points on the graph with x-coordinate 0, so the graph does not assign a *unique* value to 0. (Indeed, we can see that there are plenty of numbers between -2 and 2 to which the graph assigns multiple values.) The other two graphs do not have a comparable problem because no vertical line intersects either of the other graphs in more than one point. Graphs that pass this *vertical line test* are the graphs of functions.

Now try Exercise 5.



[–4.7, 4.7] by [–3.3, 3.3]
(a)

[–4.7, 4.7] by [–3.3, 3.3]
(b)

[–4.7, 4.7] by [–3.3, 3.3]
(c)

FIGURE 1.11 One of these is not the graph of a function. (Example 2)

Vertical Line Test

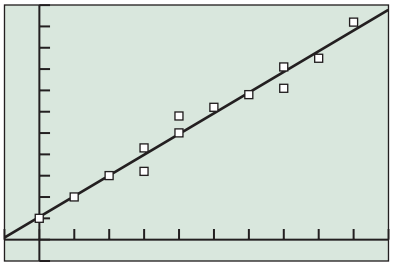
A graph (set of points (x, y)) in the xy-plane defines y as a function of x if and only if no vertical line intersects the graph in more than one point.

Domain and Range

We will usually define functions algebraically, giving the rule explicitly in terms of the domain variable. The rule, however, does not tell the complete story without some consideration of what the domain actually is.

What About Data?

When moving from a numerical model to an algebraic model we will often use a function to approximate data pairs that by themselves violate our definition. In Figure 1.12 we can see that several pairs of data points fail the vertical line test, and yet the linear function approximates the data quite well.



[–1, 10] by [–1, 11]

FIGURE 1.12 The data points fail the vertical line test but are nicely approximated by a linear function.

For example, we can define the volume of a sphere as a function of its radius by the formula

$$V(r) = \frac{4}{3}\pi r^3 \text{ (Note that this is "}V\text{ of }r\text{"—not "}V \cdot r\text{").}$$

This *formula* is defined for all real numbers, but the volume *function* is not defined for negative r-values. So, if our intention were to study the volume function, we would restrict the domain to be all $r \geq 0$.

Agreement

Unless we are dealing with a model (like volume) that necessitates a restricted domain, we will assume that the domain of a function defined by an algebraic expression is the same as the domain of the algebraic expression, the **implied domain**. For models, we will use a domain that fits the situation, the **relevant domain**.

EXAMPLE 3 Finding the Domain of a Function

Find the domain of each of these functions:

(a) $f(x) = \sqrt{x + 3}$

(b) $g(x) = \dfrac{\sqrt{x}}{x - 5}$

(c) $A(s) = (\sqrt{3}/4)s^2$, where $A(s)$ is the area of an equilateral triangle with sides of length s.

SOLUTION

Solve Algebraically

(a) The expression under a radical may not be negative. We set $x + 3 \geq 0$ and solve to find $x \geq -3$. The domain of f is the interval $[-3, \infty)$.

(b) The expression under a radical may not be negative; therefore $x \geq 0$. Also, the denominator of a fraction may not be zero; therefore $x \neq 5$. The domain of g is the interval $[0, \infty)$ with the number 5 removed, which we can write as the *union* of two intervals: $[0, 5) \cup (5, \infty)$.

(c) The algebraic expression has domain all real numbers, but the behavior being modeled restricts s from being negative. The domain of A is the interval $[0, \infty)$.

Note

The symbol "∪" is read "union." It means that the elements of the two sets are combined to form one set.

Support Graphically

We can support our answers in (a) and (b) graphically, as the calculator should not plot points where the function is undefined.

(a) Notice that the graph of $y = \sqrt{x + 3}$ (Figure 1.13a) shows points only for $x \geq -3$, as expected.

(b) The graph of $y = \sqrt{x}/(x - 5)$ (Figure 1.13b) shows points only for $x \geq 0$, as expected. Some calculators might show an unexpected line through the x-axis at $x = 5$. This line, another form of grapher failure, should not be there. Ignoring it, we see that 5, as expected, is not in the domain.

(c) The graph of $y = (\sqrt{3}/4)s^2$ (Figure 1.13c) shows the unrestricted domain of the algebraic expression: all real numbers. The calculator has no way of knowing that s is the length of a side of a triangle. ***Now try Exercise 11.***

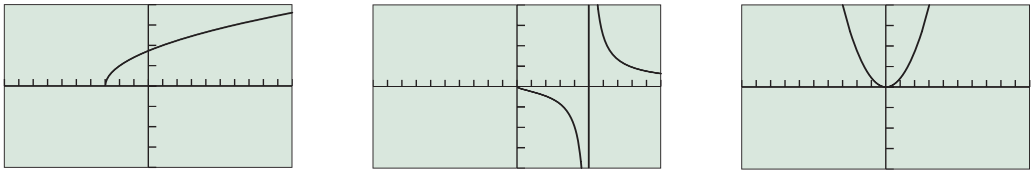


FIGURE 1.13 Graphical support of the algebraic solutions in Example 3. The vertical line in (b) should be ignored because it results from grapher failure. The points in (c) with negative *x*-coordinates should be ignored because the calculator does not know that *x* is a length (but we do).

Finding the range of a function algebraically is often much harder than finding the domain, although graphically the things we look for are similar: To find the *domain* we look for all *x-coordinates* that correspond to points on the graph, and to find the *range* we look for all *y-coordinates* that correspond to points on the graph. A good approach is to use graphical and algebraic approaches simultaneously, as we show in Example 4.

EXAMPLE 4 Finding the Range of a Function

Find the range of the function $f(x) = \frac{2}{x}$.

SOLUTION

Solve Graphically

The graph of $y = \frac{2}{x}$ is shown in Figure 1.14.

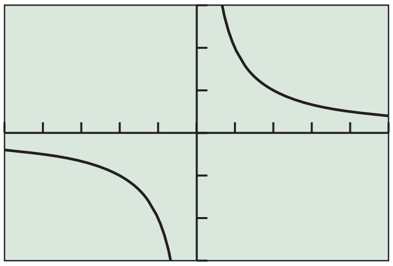


FIGURE 1.14 The graph of $y = 2/x$. Is $y = 0$ in the range?

Function Notation

A grapher typically does not use function notation. So the function $f(x) = x^2 + 1$ is entered as $y_1 = x^2 + 1$. On some graphers you can evaluate f at $x = 3$ by entering $y_1(3)$ on the home screen. On the other hand, on other graphers $y_1(3)$ means $y_1 * 3$.

It appears that $x = 0$ is not in the domain (as expected, because a denominator cannot be zero). It also appears that the range consists of all real numbers except 0.

Confirm Algebraically

We confirm that 0 is not in the range by trying to solve $2/x = 0$:

$$\frac{2}{x} = 0$$
$$2 = 0 \cdot x$$
$$2 = 0$$

(continued)

Since the equation $2 = 0$ is never true, $2/x = 0$ has no solutions, and so $y = 0$ is not in the range. But how do we know that all other real numbers are in the range? We let k be any other real number and try to solve $2/x = k$:

$$\frac{2}{x} = k$$
$$2 = k \cdot x$$
$$x = \frac{2}{k}$$

As you can see, there was no problem finding an x this time, so 0 is the only number not in the range of f. We write the range $(-\infty, 0) \cup (0, \infty)$.

Now try Exercise 17.

You can see that this is considerably more involved than finding a domain, but we are hampered at this point by not having many tools with which to analyze function behavior. We will revisit the problem of finding ranges in Exercise 86, after having developed the tools that will simplify the analysis.

Continuity

One of the most important properties of the majority of functions that model real-world behavior is that they are *continuous*. Graphically speaking, a function is continuous at a point if the graph does not come apart at that point. We can illustrate the concept with a few graphs (Figure 1.15):

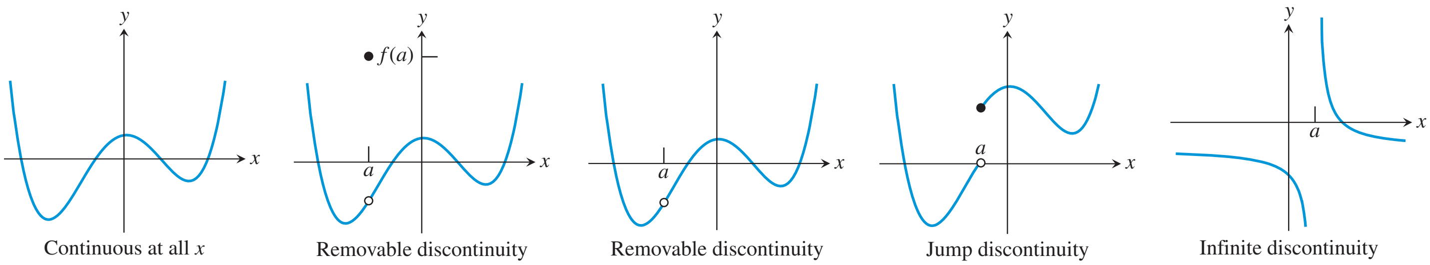
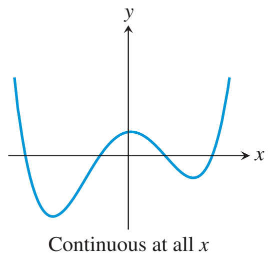


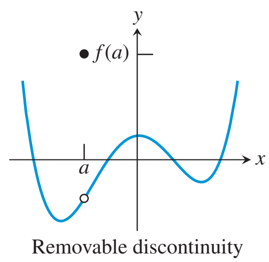
FIGURE 1.15 Some points of discontinuity.

Let's look at these cases individually.

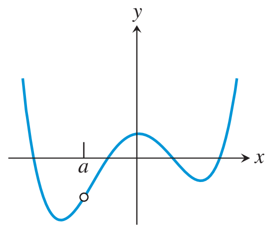
This graph is continuous everywhere. Notice that the graph has no breaks. This means that if we are studying the behavior of the function f for x-values close to any particular real number a, we can be assured that the $f(x)$-values will be close to $f(a)$.



This graph is continuous everywhere except for the "hole" at $x = a$. If we are studying the behavior of this function f for x-values close to a, we *cannot* be assured that the $f(x)$-values will be close to $f(a)$. In this case, $f(x)$ is smaller than $f(a)$ for x near a. This is called a **removable discontinuity** because it can be patched by redefining $f(a)$ so as to plug the hole.

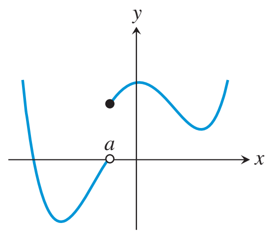


This graph also has a **removable discontinuity** at $x = a$. If we are studying the behavior of this function f for x-values close to a, we are still not assured that the $f(x)$-values will be close to $f(a)$, because in this case $f(a)$ doesn't even exist. It is removable because we could define $f(a)$ in such a way as to plug the hole and make f continuous at a.



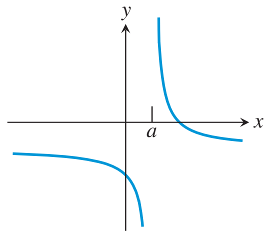
Removable discontinuity

Here is a discontinuity that is not removable. It is a **jump discontinuity** because there is more than just a hole at $x = a$; there is a *jump* in function values that makes the gap impossible to plug with a single point $(a, f(a))$, no matter how we try to redefine $f(a)$.



Jump discontinuity

This is a function with an **infinite discontinuity** at $x = a$. It is definitely not removable.



Infinite discontinuity

A Limited Use of Limits

While the *notation* of limits is easy to understand, the algebraic *definition* of a limit can be a little intimidating and is best left to future courses. We will have more to say about limits in Chapter 10. For now, if you understand the statement $\lim_{x \to 5}(x^2 - 1) = 24$, you are where you need to be.

The simple geometric concept of an unbroken graph at a point is a visual notion that is extremely difficult to communicate accurately in the language of algebra. The key concept from the pictures seems to be that we want the point $(x, f(x))$ to slide smoothly onto the point $(a, f(a))$ without missing it from either direction. We say that $f(x)$ approaches $f(a)$ as a *limit* as x approaches a, and we write $\lim_{x \to a} f(x) = f(a)$. This "limit notation" reflects graphical behavior so naturally that we will use it throughout this book as an efficient way to describe function behavior, beginning with this definition of continuity. A function f is **continuous at $x = a$** if $\lim_{x \to a} f(x) = f(a)$. A function is **discontinuous at $x = a$** if it is not continuous at $x = a$.

EXAMPLE 5 Identifying Points of Discontinuity

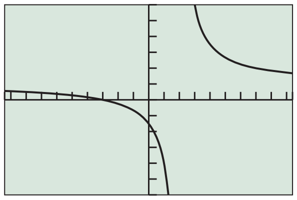
Judging from the graphs, which of the following figures shows functions that are discontinuous at $x = 2$? Are any of the discontinuities removable?

SOLUTION Figure 1.16 shows a function that is undefined at $x = 2$ and hence not continuous there. The discontinuity at $x = 2$ is not removable.

The function graphed in Figure 1.17 is a quadratic polynomial whose graph is a parabola, a graph that has no breaks because its domain includes all real numbers. It is continuous for all x.

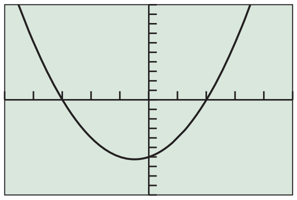
The function graphed in Figure 1.18 is not defined at $x = 2$ and so cannot be continuous there. The graph looks like the graph of the line $y = x + 2$, except that there is a hole where the point $(2, 4)$ should be. This is a removable discontinuity.

Now try Exercise 21.



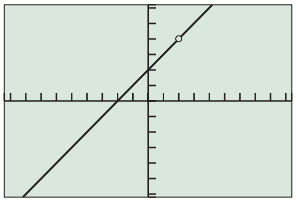
[–9.4, 9.4] by [–6, 6]

FIGURE 1.16 $f(x) = \dfrac{x + 3}{x - 2}$



[–5, 5] by [–10, 10]

FIGURE 1.17 $g(x) = (x + 3)(x - 2)$

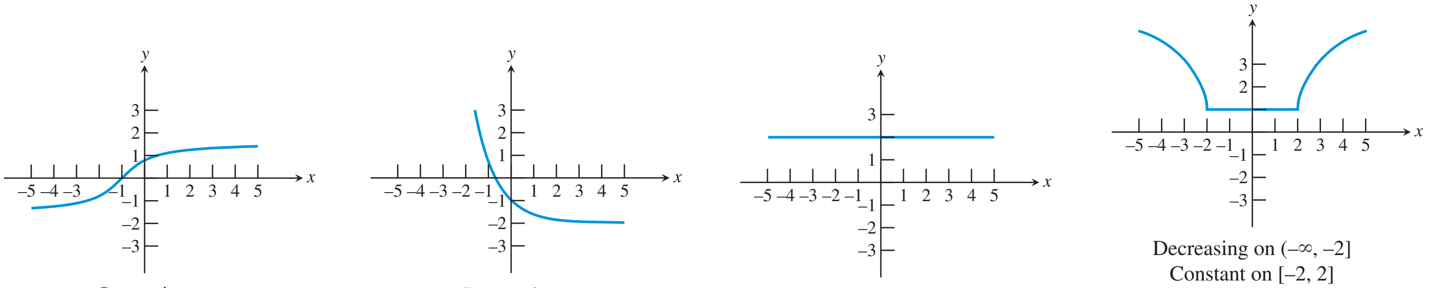


[–9.4, 9.4] by [–6.2, 6.2]

FIGURE 1.18 $h(x) = \dfrac{x^2 - 4}{x - 2}$

Increasing and Decreasing Functions

Another function concept that is easy to understand graphically is the property of being increasing, decreasing, or constant on an interval. We illustrate the concept with a few graphs (Figure 1.19):



Increasing

Decreasing

Constant

Decreasing on $(-\infty, -2]$
Constant on $[-2, 2]$
Increasing on $[2, \infty)$

FIGURE 1.19 Examples of increasing, decreasing, or constant on an interval.

Once again the idea is easy to communicate graphically, but how can we identify these properties of functions algebraically? Exploration 1 will help to set the stage for the algebraic definition.

EXPLORATION 1 Increasing, Decreasing, and Constant Data

1. Of the three tables of numerical data below, which would be modeled by a function that is (a) increasing, (b) decreasing, (c) constant?

X	Y1
−2	12
−1	12
0	12
1	12
3	12
7	12

X	Y2
−2	3
−1	1
0	0
1	−2
3	−6
7	−12

X	Y3
−2	−5
−1	−3
0	−1
1	1
3	4
7	10

ΔList on a Calculator

Your calculator might be able to help you with the numbers in Exploration 1. Some calculators have a "ΔList" operation that will calculate the changes as you move down a list. For example, the command "ΔList (L1) → L3" will store the differences from L1 into L3. Note that ΔList (L1) is always one entry shorter than L1 itself.

2. Make a list of ΔY1, the *change* in Y1 values as you move down the list. As you move from Y1 $= a$ to Y1 $= b$, the change is ΔY1 $= b - a$. Do the same for the values of Y2 and Y3.

X moves from	ΔX	ΔY1
−2 to −1	1	
−1 to 0	1	
0 to 1	1	
1 to 3	2	
3 to 7	4	

X moves from	ΔX	ΔY2
−2 to −1	1	
−1 to 0	1	
0 to 1	1	
1 to 3	2	
3 to 7	4	

X moves from	ΔX	ΔY3
−2 to −1	1	
−1 to 0	1	
0 to 1	1	
1 to 3	2	
3 to 7	4	

3. What is true about the quotients ΔY/ΔX for an increasing function? For a decreasing function? For a constant function?
4. Where else have you seen the quotient ΔY/ΔX? Does this reinforce your answers in part 3?

Your analysis of the quotients ΔY/ΔX in the exploration should help you to understand the following definition.

DEFINITION Increasing, Decreasing, and Constant Function on an Interval

A function f is **increasing** on an interval if, for any two points in the interval, a positive change in x results in a positive change in $f(x)$.

A function f is **decreasing** on an interval if, for any two points in the interval, a positive change in x results in a negative change in $f(x)$.

A function f is **constant** on an interval if, for any two points in the interval, a positive change in x results in a zero change in $f(x)$.

EXAMPLE 6 Analyzing a Function for Increasing-Decreasing Behavior

For each function, tell the intervals on which it is increasing and the intervals on which it is decreasing.

(a) $f(x) = (x + 2)^2$ **(b)** $g(x) = \dfrac{x^2}{x^2 - 1}$

SOLUTION

Solve Graphically

(a) We see from the graph in Figure 1.20 that f is decreasing on $(-\infty, -2]$ and increasing on $[-2, \infty)$. (Notice that we include −2 in both intervals. Don't worry that this sets up some contradiction about what happens *at* −2, because we only talk about functions increasing or decreasing on *intervals*, and −2 is not an interval.)

(continued)

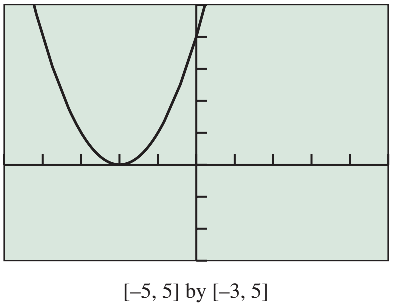


FIGURE 1.20 The function $f(x) = (x + 2)^2$ decreases on $(-\infty, -2]$ and increases on $[-2, \infty)$. (Example 6)

(b) We see from the graph in Figure 1.21 that g is increasing on $(-\infty, -1)$, increasing again on $(-1, 0]$, decreasing on $[0, 1)$, and decreasing again on $(1, \infty)$.

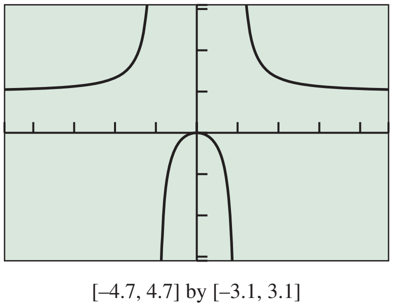


FIGURE 1.21 The function $g(x) = x^2/(x^2 - 1)$ increases on $(-\infty, -1)$ and $(-1, 0]$; the function decreases on $[0, 1)$ and $(1, \infty)$. (Example 6)

Now try Exercise 33.

You may have noticed that we are making some assumptions about the graphs. How do we know that they don't turn around somewhere off the screen? We will develop some ways to answer that question later in the book, but the most powerful methods will await you when you study calculus.

Boundedness

The concept of *boundedness* is fairly simple to understand both graphically and algebraically. We will move directly to the algebraic definition after motivating the concept with some typical graphs (Figure 1.22).

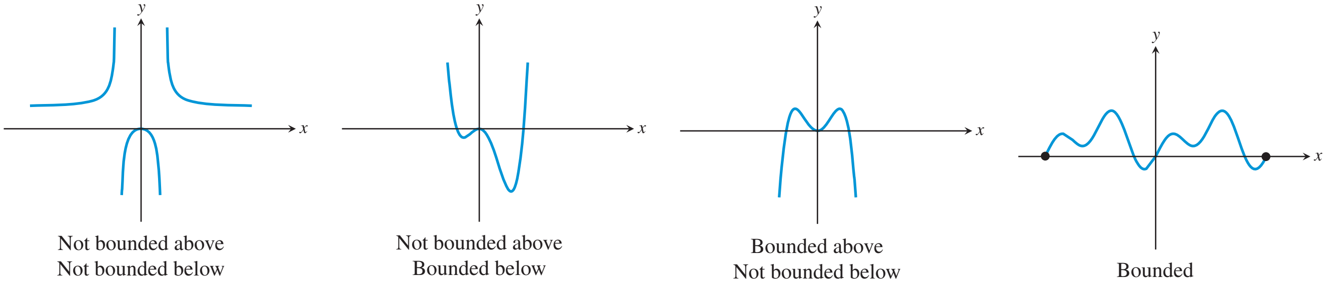


FIGURE 1.22 Some examples of graphs bounded and not bounded above and below.

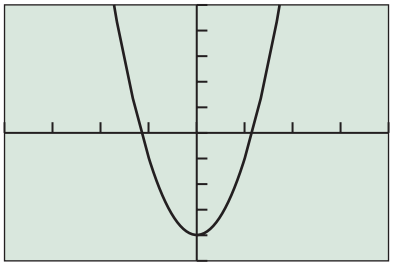
DEFINITION Lower Bound, Upper Bound, and Bounded

A function f is **bounded below** if there is some number b that is less than or equal to every number in the range of f. Any such number b is called a **lower bound** of f.

A function f is **bounded above** if there is some number B that is greater than or equal to every number in the range of f. Any such number B is called an **upper bound** of f.

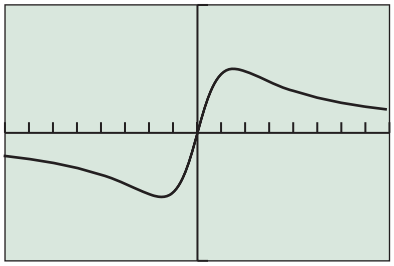
A function f is **bounded** if it is bounded both above and below.

We can extend the above definition to the idea of **bounded on an interval** by restricting the domain of consideration in each part of the definition to the interval we wish to consider. For example, the function $f(x) = 1/x$ is bounded above on the interval $(-\infty, 0)$ and bounded below on the interval $(0, \infty)$.



[–4, 4] by [–5, 5]

(a)



[–8, 8] by [–1, 1]

(b)

FIGURE 1.23 The graphs for Example 7. Which are bounded where?

EXAMPLE 7 Checking Boundedness

Identify each of these functions as bounded below, bounded above, or bounded.

(a) $w(x) = 3x^2 - 4$ **(b)** $p(x) = \dfrac{x}{1 + x^2}$

SOLUTION

Solve Graphically

The two graphs are shown in Figure 1.23. It appears that w is bounded below, and p is bounded.

Confirm Graphically

We can confirm that w is bounded below by finding a lower bound, as follows:

$$x^2 \geq 0 \qquad x^2 \text{ is nonnegative.}$$
$$3x^2 \geq 0 \qquad \text{Multiply by 3.}$$
$$3x^2 - 4 \geq 0 - 4 \qquad \text{Subtract 4.}$$
$$3x^2 - 4 \geq -4$$

Thus, -4 is a lower bound for $w(x) = 3x^2 - 4$.

We leave the verification that p is bounded as an exercise (Exercise 77).

Now try Exercise 37.

Local and Absolute Extrema

Many graphs are characterized by peaks and valleys where they change from increasing to decreasing and vice versa. The extreme values of the function (or *local extrema*) can be characterized as either *local maxima* or *local minima*. The distinction can be easily seen graphically. Figure 1.24 shows a graph with three local extrema: local maxima at points P and R and a local minimum at Q.

This is another function concept that is easier to see graphically than to describe algebraically. Notice that a local maximum does not have to be *the* maximum value of a function; it only needs to be the maximum value of the function on *some* tiny interval.

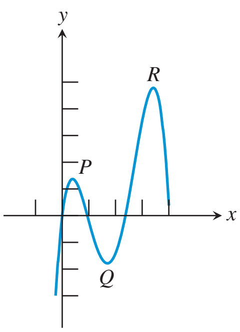


FIGURE 1.24 The graph suggests that f has a local maximum at P, a local minimum at Q, and a local maximum at R.

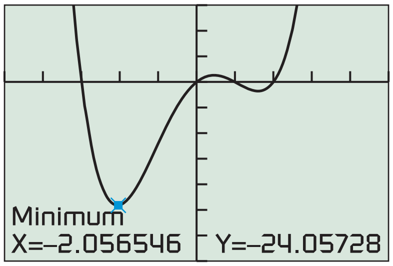
We have already mentioned that the best method for analyzing increasing and decreasing behavior involves calculus. Not surprisingly, the same is true for local extrema. We will generally be satisfied in this course with approximating local extrema using a graphing calculator, although sometimes an algebraic confirmation will be possible when we learn more about specific functions.

DEFINITION Local and Absolute Extrema

A **local maximum** of a function f is a value $f(c)$ that is greater than or equal to all range values of f on some open interval containing c. If $f(c)$ is greater than or equal to all range values of f, then $f(c)$ is the **maximum** (or **absolute maximum**) value of f.

A **local minimum** of a function f is a value $f(c)$ that is less than or equal to all range values of f on some open interval containing c. If $f(c)$ is less than or equal to all range values of f, then $f(c)$ is the **minimum** (or **absolute minimum**) value of f.

Local extrema are also called **relative extrema**.



[–5, 5] by [–35, 15]

FIGURE 1.25 A graph of $y = x^4 - 7x^2 + 6x$. (Example 8)

EXAMPLE 8 Identifying Local Extrema

Decide whether $f(x) = x^4 - 7x^2 + 6x$ has any local maxima or local minima. If so, find each local maximum value or minimum value and the value of x at which each occurs.

SOLUTION The graph of $y = x^4 - 7x^2 + 6x$ (Figure 1.25) suggests that there are two local minimum values and one local maximum value. We use the graphing calculator to approximate local minima as -24.06 (which occurs at $x \approx -2.06$) and -1.77 (which occurs at $x \approx 1.60$). Similarly, we identify the (approximate) local maximum as 1.32 (which occurs at $x \approx 0.46$).

Now try Exercise 41.

Using a Grapher to Find Local Extrema

Most modern graphers have built-in "maximum" and "minimum" finders that identify local extrema by looking for sign changes in Δy. It is not easy to find local extrema by zooming in on them, as the graphs tend to flatten out and hide the very behavior you are looking at. If you use this method, keep narrowing the vertical window to maintain some curve in the graph.

Symmetry

In the graphical sense, the word "symmetry" in mathematics carries essentially the same meaning as it does in art: The picture (in this case, the graph) "looks the same" when viewed in more than one way. The interesting thing about mathematical symmetry is that it can be characterized numerically and algebraically as well. We will be looking at three particular types of symmetry, each of which can be spotted easily from a graph, a table of values, or an algebraic formula, once you know what to look for. Since it is the connections among the three models (graphical, numerical, and algebraic) that we need to emphasize in this section, we will illustrate the various symmetries in all three ways, side-by-side.

Symmetry with respect to the *y*-axis
Example: $f(x) = x^2$

Graphically

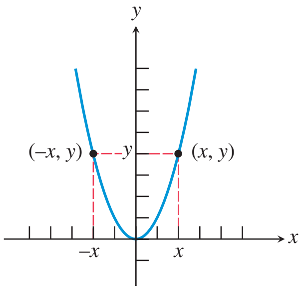


FIGURE 1.26 The graph looks the same to the left of the y-axis as it does to the right of it.

Numerically

x	$f(x)$
-3	9
-2	4
-1	1
1	1
2	4
3	9

Algebraically

For all x in the domain of f,

$$f(-x) = f(x).$$

Functions with this property (for example, x^n, n even) are **even** functions.

Symmetry with respect to the x-axis
Example: $x = y^2$

Graphically

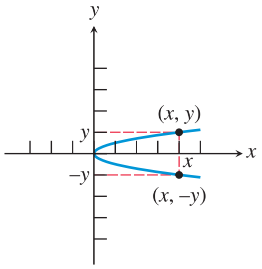


FIGURE 1.27 The graph looks the same above the x-axis as it does below it.

Numerically

x	y
9	−3
4	−2
1	−1
1	1
4	2
9	3

Algebraically

Graphs with this kind of symmetry are not functions (except the zero function), but we can say that $(x, -y)$ is on the graph whenever (x, y) is on the graph.

Symmetry with respect to the origin
Example: $f(x) = x^3$

Graphically

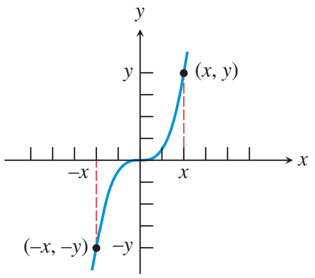


FIGURE 1.28 The graph looks the same upside-down as it does rightside-up.

Numerically

x	y
−3	−27
−2	−8
−1	−1
1	1
2	8
3	27

Algebraically

For all x in the domain of f,

$$f(-x) = -f(x).$$

Functions with this property (for example, x^n, n odd) are **odd** functions.

EXAMPLE 9 Checking Functions for Symmetry

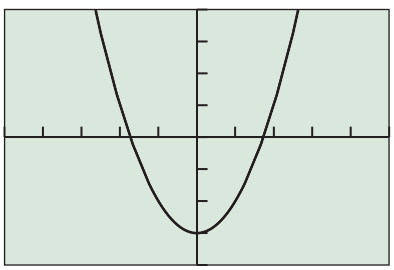
Tell whether each of the following functions is odd, even, or neither.

(a) $f(x) = x^2 - 3$ **(b)** $g(x) = x^2 - 2x - 2$ **(c)** $h(x) = \dfrac{x^3}{4 - x^2}$

SOLUTION

(a) Solve Graphically

The graphical solution is shown in Figure 1.29.



[–5, 5] by [–4, 4]

FIGURE 1.29 This graph appears to be symmetric with respect to the y-axis, so we conjecture that f is an even function.

(continued)

Confirm Algebraically

We need to verify that

$$f(-x) = f(x)$$

for all x in the domain of f.

$$f(-x) = (-x)^2 - 3 = x^2 - 3$$
$$= f(x)$$

Since this identity is true for all x, the function f is indeed even.

(b) Solve Graphically

The graphical solution is shown in Figure 1.30.

Confirm Algebraically

We need to verify that

$$g(-x) \neq g(x) \text{ and } g(-x) \neq -g(x).$$
$$g(-x) = (-x)^2 - 2(-x) - 2$$
$$= x^2 + 2x - 2$$
$$g(x) = x^2 - 2x - 2$$
$$-g(x) = -x^2 + 2x + 2$$

So $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$.

We conclude that g is neither odd nor even.

(c) Solve Graphically

The graphical solution is shown in Figure 1.31.

Confirm Algebraically

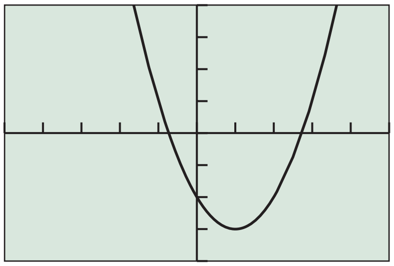
We need to verify that

$$h(-x) = -h(x)$$

for all x in the domain of h.

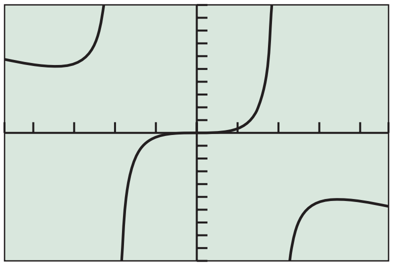
$$h(-x) = \frac{(-x)^3}{4 - (-x)^2} = \frac{-x^3}{4 - x^2}$$
$$= -h(x)$$

Since this identity is true for all x except ± 2 (which are not in the domain of h), the function h is odd. ***Now try Exercise 49.***



[–5, 5] by [–4, 4]

FIGURE 1.30 This graph does not appear to be symmetric with respect to either the y-axis or the origin, so we conjecture that g is neither even nor odd.



[–4.7, 4.7] by [–10, 10]

FIGURE 1.31 This graph appears to be symmetric with respect to the origin, so we conjecture that h is an odd function.

Asymptotes

Consider the graph of the function $f(x) = \dfrac{2x^2}{4 - x^2}$ in Figure 1.32.

The graph appears to flatten out to the right and to the left, getting closer and closer to the horizontal line $y = -2$. We call this line a *horizontal asymptote*. Similarly, the graph appears to flatten out as it goes off the top and bottom of the screen, getting closer and closer to the vertical lines $x = -2$ and $x = 2$. We call these lines *vertical asymptotes*. If we superimpose the asymptotes onto Figure 1.32 as dashed lines, you can see that they form a kind of template that describes the limiting behavior of the graph (Figure 1.33 on the next page).

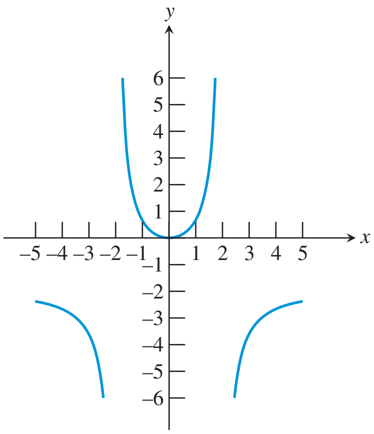


FIGURE 1.32 The graph of $f(x) = 2x^2/(4 - x^2)$ has two vertical asymptotes and one horizontal asymptote.

Since asymptotes describe the behavior of the graph at its horizontal or vertical extremities, the definition of an asymptote can best be stated with limit notation. In this definition, note that $x \to a^-$ means "x approaches a from the left," while $x \to a^+$ means "x approaches a from the right."

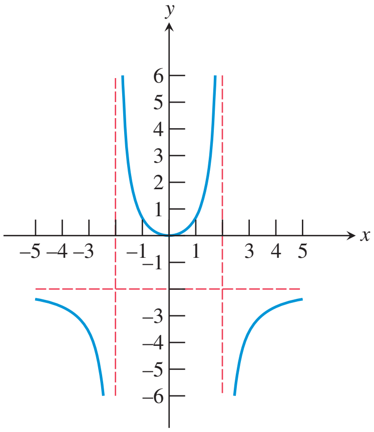
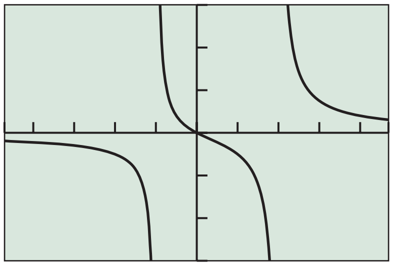


FIGURE 1.33 The graph of $f(x) = 2x^2/(4 - x^2)$ with the asymptotes shown as dashed lines.



[–4.7, 4.7] by [–3, 3]

FIGURE 1.34 The graph of $y = x/(x^2 - x - 2)$ has vertical asymptotes of $x = -1$ and $x = 2$ and a horizontal asymptote of $y = 0$. (Example 10)

DEFINITION Horizontal and Vertical Asymptotes

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if $f(x)$ approaches a limit of b as x approaches $+\infty$ or $-\infty$.

In limit notation:

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to +\infty} f(x) = b$$

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if $f(x)$ approaches a limit of $+\infty$ or $-\infty$ as x approaches a from either direction.

In limit notation:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

EXAMPLE 10 Identifying the Asymptotes of a Graph

Identify any horizontal or vertical asymptotes of the graph of

$$y = \frac{x}{x^2 - x - 2}.$$

SOLUTION The quotient $x/(x^2 - x - 2) = x/((x + 1)(x - 2))$ is undefined at $x = -1$ and $x = 2$, which makes them likely sites for vertical asymptotes. The graph (Figure 1.34) provides support, showing vertical asymptotes of $x = -1$ and $x = 2$.

For large values of x, the numerator (a large number) is dwarfed by the denominator (a *product* of *two* large numbers), suggesting that $\lim_{x \to \infty} x/((x + 1)(x - 2)) = 0$. This would indicate a horizontal asymptote of $y = 0$. The graph (Figure 1.34) provides support, showing a horizontal asymptote of $y = 0$ as $x \to \infty$. Similar logic suggests that $\lim_{x \to -\infty} x/((x + 1)(x - 2)) = -0 = 0$, indicating the same horizontal asymptote as $x \to -\infty$. Again, the graph provides support for this.

Now try Exercise 57.

End Behavior

A horizontal asymptote gives one kind of end behavior for a function because it shows how the function behaves as it goes off toward either "end" of the x-axis. Not all graphs approach lines, but it is helpful to consider what *does* happen "out there." We illustrate with a few examples.

EXAMPLE 11 Matching Functions Using End Behavior

Match the functions with the graphs in Figure 1.35 by considering end behavior. All graphs are shown in the same viewing window.

(a) $y = \dfrac{3x}{x^2 + 1}$ **(b)** $y = \dfrac{3x^2}{x^2 + 1}$

(c) $y = \dfrac{3x^3}{x^2 + 1}$ **(d)** $y = \dfrac{3x^4}{x^2 + 1}$

(continued)

Tips on Zooming

Zooming out is often a good way to investigate end behavior with a graphing calculator. Here are some useful zooming tips:

- Start with a "square" window.
- Set Xscl and Yscl to zero to avoid fuzzy axes.
- Be sure the zoom factors are both the same. (They will be unless you change them.)

SOLUTION When x is very large, the denominator $x^2 + 1$ in each of these functions is almost the same number as x^2. If we replace $x^2 + 1$ in each denominator by x^2 and then reduce the fractions, we get the simpler functions

(a) $y = \dfrac{3}{x}$ (close to $y = 0$ for large x) **(b)** $y = 3$

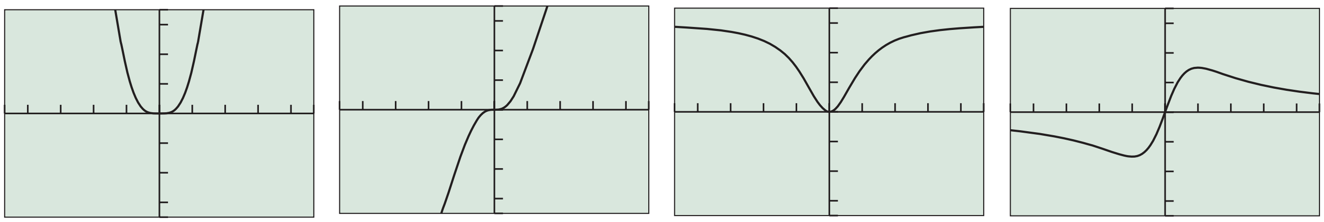
(c) $y = 3x$ **(d)** $y = 3x^2$.

So, we look for functions that have end behavior resembling, respectively, the functions

(a) $y = 0$ **(b)** $y = 3$ **(c)** $y = 3x$ **(d)** $y = 3x^2$.

Graph (iv) approaches the line $y = 0$. Graph (iii) approaches the line $y = 3$. Graph (ii) approaches the line $y = 3x$. Graph (i) approaches the parabola $y = 3x^2$. So, (a) matches (iv), (b) matches (iii), (c) matches (ii), and (d) matches (i).

Now try Exercise 65.



[–4.7, 4.7] by [–3.5, 3.5] (i) [–4.7, 4.7] by [–3.5, 3.5] (ii) [–4.7, 4.7] by [–3.5, 3.5] (iii) [–4.7, 4.7] by [–3.5, 3.5] (iv)

FIGURE 1.35 Match the graphs with the functions in Example 11.

For more complicated functions we are often content with knowing whether the end behavior is bounded or unbounded in either direction.

QUICK REVIEW 1.2 *(For help, go to Sections A.3, P.3, and P.5.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1–4, solve the equation or inequality.

1. $x^2 - 16 = 0$ **2.** $9 - x^2 = 0$

3. $x - 10 < 0$ **4.** $5 - x \le 0$

In Exercises 5–10, find all values of x algebraically for which the algebraic expression is *not* defined. Support your answer graphically.

5. $\dfrac{x}{x - 16}$ **6.** $\dfrac{x}{x^2 - 16}$

7. $\sqrt{x - 16}$ **8.** $\dfrac{\sqrt{x^2 + 1}}{x^2 - 1}$

9. $\dfrac{\sqrt{x + 2}}{\sqrt{3 - x}}$

10. $\dfrac{x^2 - 2x}{x^2 - 4}$

SECTION 1.2 EXERCISES

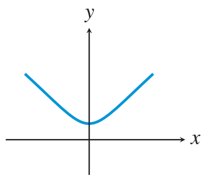
In Exercises 1–4, determine whether the formula determines y as a function of x. If not, explain why not.

1. $y = \sqrt{x - 4}$ **2.** $y = x^2 \pm 3$

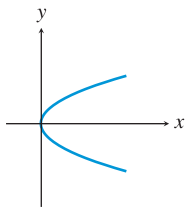
3. $x = 2y^2$ **4.** $x = 12 - y$

In Exercises 5–8, use the vertical line test to determine whether the curve is the graph of a function.

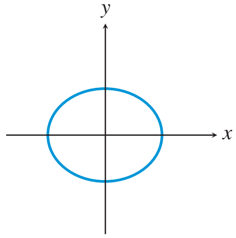
5.



6.



7.



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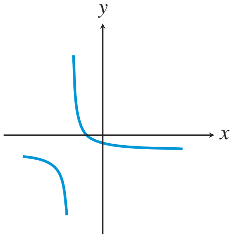


In Exercises 9–16, find the domain of the function algebraically and support your answer graphically.

9. $f(x) = x^2 + 4$

10. $h(x) = \dfrac{5}{x - 3}$

11. $f(x) = \dfrac{3x - 1}{(x + 3)(x - 1)}$

12. $f(x) = \dfrac{1}{x} + \dfrac{5}{x - 3}$

13. $g(x) = \dfrac{x}{x^2 - 5x}$

14. $h(x) = \dfrac{\sqrt{4 - x^2}}{x - 3}$

15. $h(x) = \dfrac{\sqrt{4 - x}}{(x + 1)(x^2 + 1)}$

16. $f(x) = \sqrt{x^4 - 16x^2}$

In Exercises 17–20, find the range of the function.

17. $f(x) = 10 - x^2$

18. $g(x) = 5 + \sqrt{4 - x}$

19. $f(x) = \dfrac{x^2}{1 - x^2}$

20. $g(x) = \dfrac{3 + x^2}{4 - x^2}$

In Exercises 21–24, graph the function and tell whether or not it has a point of discontinuity at $x = 0$. If there is a discontinuity, tell whether it is removable or nonremovable.

21. $g(x) = \dfrac{3}{x}$

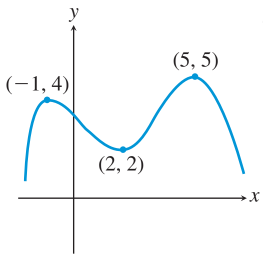
22. $h(x) = \dfrac{x^3 + x}{x}$

23. $f(x) = \dfrac{|x|}{x}$

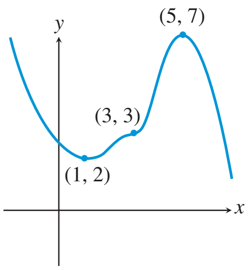
24. $g(x) = \dfrac{x}{x - 2}$

In Exercises 25–28, state whether each labeled point identifies a local minimum, a local maximum, or neither. Identify intervals on which the function is decreasing and increasing.

25.



26.



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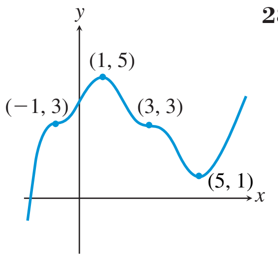
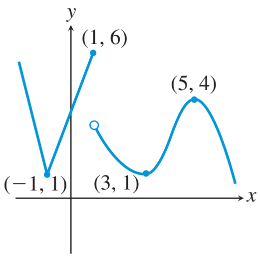


28.



In Exercises 29–34, graph the function and identify intervals on which the function is increasing, decreasing, or constant.

29. $f(x) = |x + 2| - 1$

30. $f(x) = |x + 1| + |x - 1| - 3$

31. $g(x) = |x + 2| + |x - 1| - 2$

32. $h(x) = 0.5(x + 2)^2 - 1$

33. $g(x) = 3 - (x - 1)^2$

34. $f(x) = x^3 - x^2 - 2x$

In Exercises 35–40, determine whether the function is bounded above, bounded below, or bounded on its domain.

35. $y = 32$

36. $y = 2 - x^2$

37. $y = 2^x$

38. $y = 2^{-x}$

39. $y = \sqrt{1 - x^2}$

40. $y = x - x^3$

In Exercises 41–46, use a grapher to find all local maxima and minima and the values of x where they occur. Give values rounded to two decimal places.

41. $f(x) = 4 - x + x^2$

42. $g(x) = x^3 - 4x + 1$

43. $h(x) = -x^3 + 2x - 3$

44. $f(x) = (x + 3)(x - 1)^2$

45. $h(x) = x^2\sqrt{x + 4}$

46. $g(x) = x|2x + 5|$

In Exercises 47–54, state whether the function is odd, even, or neither. Support graphically and confirm algebraically.

47. $f(x) = 2x^4$

48. $g(x) = x^3$

49. $f(x) = \sqrt{x^2 + 2}$

50. $g(x) = \dfrac{3}{1 + x^2}$

51. $f(x) = -x^2 + 0.03x + 5$

52. $f(x) = x^3 + 0.04x^2 + 3$

53. $g(x) = 2x^3 - 3x$

54. $h(x) = \dfrac{1}{x}$

In Exercises 55–62, use a method of your choice to find all horizontal and vertical asymptotes of the function.

55. $f(x) = \dfrac{x}{x - 1}$

56. $q(x) = \dfrac{x - 1}{x}$

57. $g(x) = \dfrac{x + 2}{3 - x}$

58. $q(x) = 1.5^x$

59. $f(x) = \dfrac{x^2 + 2}{x^2 - 1}$

60. $p(x) = \dfrac{4}{x^2 + 1}$

61. $g(x) = \dfrac{4x - 4}{x^3 - 8}$

62. $h(x) = \dfrac{2x - 4}{x^2 - 4}$

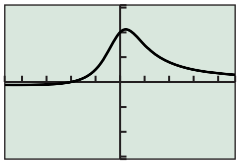
In Exercises 63–66, match the function with the corresponding graph by considering end behavior and asymptotes. All graphs are shown in the same viewing window.

63. $y = \dfrac{x + 2}{2x + 1}$

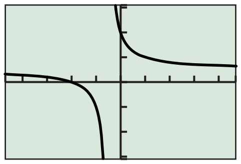
64. $y = \dfrac{x^2 + 2}{2x + 1}$

65. $y = \dfrac{x + 2}{2x^2 + 1}$

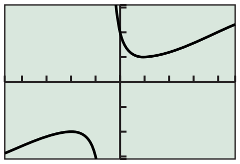
66. $y = \dfrac{x^3 + 2}{2x^2 + 1}$



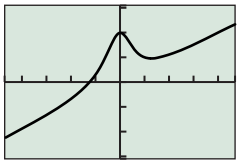
[–4.7, 4.7] by [–3.1, 3.1]
(a)



[–4.7, 4.7] by [–3.1, 3.1]
(b)



[–4.7, 4.7] by [–3.1, 3.1]
(c)



[–4.7, 4.7] by [–3.1, 3.1]
(d)

67. Can a Graph Cross Its Own Asymptote? The Greek roots of the word "asymptote" mean "not meeting," since graphs tend to approach, but not meet, their asymptotes. Which of the following functions have graphs that *do* intersect their horizontal asymptotes?

(a) $f(x) = \dfrac{x}{x^2 - 1}$

(b) $g(x) = \dfrac{x}{x^2 + 1}$

(c) $h(x) = \dfrac{x^2}{x^3 + 1}$

68. Can a Graph Have Two Horizontal Asymptotes? Although most graphs have at most one horizontal asymptote, it is possible for a graph to have more than one. Which of the following functions have graphs with more than one horizontal asymptote?

(a) $f(x) = \dfrac{|x^3 + 1|}{8 - x^3}$

(b) $g(x) = \dfrac{|x - 1|}{x^2 - 4}$

(c) $h(x) = \dfrac{x}{\sqrt{x^2 - 4}}$

69. Can a Graph Intersect Its Own Vertical Asymptote? Graph the function $f(x) = \dfrac{x - |x|}{x^2} + 1$.

(a) The graph of this function does not intersect its vertical asymptote. Explain why it does not.

(b) Show how you can add a single point to the graph of f and get a graph that *does* intersect its vertical asymptote.

(c) Is the graph in (b) the graph of a function?

70. Writing to Learn Explain why a graph cannot have more than two horizontal asymptotes.

Standardized Test Questions

71. True or False The graph of function f is defined as the set of all points $(x, f(x))$, where x is in the domain of f. Justify your answer.

72. True or False A relation that is symmetric with respect to the x-axis cannot be a function. Justify your answer.

In Exercises 73–76, answer the question without using a calculator.

73. Multiple Choice Which function is continuous?

(A) Number of children enrolled in a particular school as a function of time

(B) Outdoor temperature as a function of time

(C) Cost of U.S. postage as a function of the weight of the letter

(D) Price of a stock as a function of time

(E) Number of soft drinks sold at a ballpark as a function of outdoor temperature

74. Multiple Choice Which function is *not* continuous?

(A) Your altitude as a function of time while flying from Reno to Dallas

(B) Time of travel from Miami to Pensacola as a function of driving speed

(C) Number of balls that can fit completely inside a particular box as a function of the radius of the balls

(D) Area of a circle as a function of radius

(E) Weight of a particular baby as a function of time after birth

75. Decreasing Function Which function is decreasing?

(A) Outdoor temperature as a function of time

(B) The Dow Jones Industrial Average as a function of time

(C) Air pressure in the Earth's atmosphere as a function of altitude

(D) World population since 1900 as a function of time

(E) Water pressure in the ocean as a function of depth

76. Increasing or Decreasing Which function cannot be classified as either increasing or decreasing?

(A) Weight of a lead brick as a function of volume

(B) Strength of a radio signal as a function of distance from the transmitter

(C) Time of travel from Buffalo to Syracuse as a function of driving speed

(D) Area of a square as a function of side length

(E) Height of a swinging pendulum as a function of time

Explorations

77. Bounded Functions As promised in Example 7 of this section, we will give you a chance to prove algebraically that $p(x) = x/(1 + x^2)$ is bounded.

(a) Graph the function and find the smallest integer k that appears to be an upper bound.

(b) Verify that $x/(1 + x^2) < k$ by proving the equivalent inequality $kx^2 - x + k > 0$. (Use the quadratic formula to show that the quadratic has no real zeros.)

(c) From the graph, find the greatest integer k that appears to be a lower bound.

(d) Verify that $x/(1 + x^2) > k$ by proving the equivalent inequality $kx^2 - x + k < 0$.

78. Baylor School Grade Point Averages Baylor School uses a sliding scale to convert the percentage grades on its transcripts to grade point averages (GPAs). Table 1.9 shows the GPA equivalents for selected grades.

Table 1.9 Converting Grades

Grade (x)	GPA (y)
60	0.00
65	1.00
70	2.05
75	2.57
80	3.00
85	3.36
90	3.69
95	4.00
100	4.28

Source: Baylor School College Counselor.

(a) Considering GPA (y) as a function of percentage grade (x), is it increasing, decreasing, constant, or none of these?

(b) Make a table showing the *change* (Δy) in GPA as you move down the list. (See Exploration 1.)

(c) Make a table showing the change in Δy as you move down the list. (This is $\Delta\Delta y$.) Considering the *change* (Δy) in GPA as a function of percentage grade (x), is it increasing, decreasing, constant, or none of these?

(d) In general, what can you say about the shape of the graph if y is an increasing function of x and Δy is a decreasing function of x?

(e) Sketch the graph of a function y of x such that y is a decreasing function of x and Δy is an increasing function of x.

79. Group Activity Sketch (freehand) a graph of a function f with domain all real numbers that satisfies all of the following conditions:

(a) f is continuous for all x;

(b) f is increasing on $(-\infty, 0]$ and on $[3, 5]$;

(c) f is decreasing on $[0, 3]$ and on $[5, \infty)$;

(d) $f(0) = f(5) = 2$;

(e) $f(3) = 0$.

80. Group Activity Sketch (freehand) a graph of a function f with domain all real numbers that satisfies all of the following conditions:

(a) f is decreasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$;

(b) f has a nonremovable point of discontinuity at $x = 0$;

(c) f has a horizontal asymptote at $y = 1$;

(d) $f(0) = 0$;

(e) f has a vertical asymptote at $x = 0$.

81. Group Activity Sketch (freehand) a graph of a function f with domain all real numbers that satisfies all of the following conditions:

(a) f is continuous for all x;

(b) f is an even function;

(c) f is increasing on $[0, 2]$ and decreasing on $[2, \infty)$;

(d) $f(2) = 3$.

82. Group Activity Get together with your classmates in groups of two or three. Sketch a graph of a function, but do not show it to the other members of your group. Using the language of functions (as in Exercises 79–81), describe your function as completely as you can. Exchange descriptions with the others in your group and see if you can reproduce each other's graphs.

Extending the Ideas

83. A function that is bounded above has an infinite number of upper bounds, but there is always a *least upper bound*, i.e., an upper bound that is less than all the others. This least upper bound may or may not be in the range of f. For each of the following functions, find the least upper bound and tell whether or not it is in the range of the function.

(a) $f(x) = 2 - 0.8x^2$

(b) $g(x) = \dfrac{3x^2}{3 + x^2}$

(c) $h(x) = \dfrac{1 - x}{x^2}$

(d) $p(x) = 2 \sin(x)$

(e) $q(x) = \dfrac{4x}{x^2 + 2x + 1}$

84. Writing to Learn A continuous function f has domain all real numbers. If $f(-1) = 5$ and $f(1) = -5$, explain why f must have at least one zero in the interval $[-1, 1]$. (This generalizes to a property of continuous functions known as the Intermediate Value Theorem.)

85. Proving a Theorem Prove that the graph of every odd function with domain all real numbers must pass through the origin.

86. Finding the Range Graph the function $f(x) = \dfrac{3x^2 - 1}{2x^2 + 1}$ in the window $[-6, 6]$ by $[-2, 2]$.

(a) What is the apparent horizontal asymptote of the graph?

(b) Based on your graph, determine the apparent range of f.

(c) Show algebraically that $-1 \le \dfrac{3x^2 - 1}{2x^2 + 1} < 1.5$ for all x, thus confirming your conjecture in part (b).

87. Looking Ahead to Calculus A key theorem in calculus, the Extreme Value Theorem, states, if a function f is continuous on a closed interval $[a, b]$ then f has both a maximum value and a minimum value on the interval. For each of the following functions, verify that the function is continuous on the given interval and find the maximum and minimum values of the function and the x values at which these extrema occur.

(a) $f(x) = x^2 - 3$, $[-2, 4]$

(b) $f(x) = \dfrac{1}{x}$, $[1, 5]$

(c) $f(x) = |x + 1| + 2$, $[-4, 1]$

(d) $f(x) = \sqrt{x^2 + 9}$, $[-4, 4]$