■ Section 1.4 Building Functions from Functions

Exploration 1

If $f = 2x - 3$ and $g = \dfrac{x+3}{2}$, then

$f \circ g = 2\left(\dfrac{x+3}{2}\right) - 3 = x + 3 - 3 = x.$

If $f = |2x + 4|$ and $g = \dfrac{(x-2)(x+2)}{2}$,

then $f \circ g = 2\left(\dfrac{(x-2)(x+2)}{2}\right) + 4$

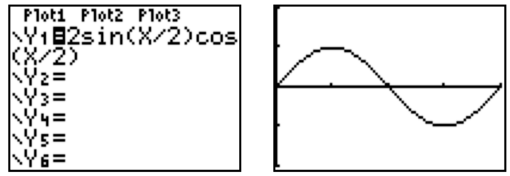
$= (x-2)(x+2) + 4 = x^2 - 4 + 4 = x^2.$

If $f = \sqrt{x}$ and $g = x^2$, then $f \circ g = \sqrt{x^2} = |x|$. Note, we use the absolute value of x because g is defined for $-\infty < x < +\infty$, while f is defined only for positive values of x. The absolute value function is always positive.

If $f = x^5$ and $g = x^{0.6}$, then $f \circ g = (x^{0.6})^5 = x^3$.

If $f = x - 3$ and $g = \ln(e^3 x)$, then $f \circ g = \ln(e^3 x) - 3 = \ln(e^3) + \ln x - 3 = 3 \ln e + \ln x - 3 = 3 + \ln x - 3 = \ln x$.

If $f = 2 \sin x \cos x$ and $g = \dfrac{x}{2}$, then $f \circ g = 2 \sin\dfrac{x}{2} \cos\dfrac{x}{2} = \sin\left(2\left(\dfrac{x}{2}\right)\right) = \sin x$. This is the double angle formula (*see Section 5.4*). You can see this graphically.

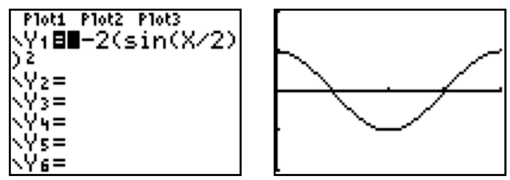


[0, 2π] by [–2, 2]

If $f = 1 - 2x^2$ and $g = \sin\left(\dfrac{x}{2}\right)$,

then $f \circ g = 1 - 2\left(\sin^2\left(\dfrac{x}{2}\right)\right) = \cos\left(2\left(\dfrac{x}{2}\right)\right) = \cos x$.

(The double angle formula for $\cos 2x$ is $\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$. *See Section 5.3.*) This can be seen graphically:



[0, 2π] by [–2, 2]

f	g	$f \circ g$
$2x - 3$	$\dfrac{x+3}{2}$	x
$\|2x + 4\|$	$\dfrac{(x-2)(x+2)}{2}$	x^2
$\sqrt{x}$	x^2	$\|x\|$
x^5	$x^{0.6}$	x^3
$x - 3$	$\ln(e^3 x)$	$\ln x$
$2 \sin x \cos x$	$\dfrac{x}{2}$	$\sin x$
$1 - 2x^2$	$\sin\left(\dfrac{x}{2}\right)$	$\cos x$

Quick Review 1.4

1. $(-\infty, -3) \cup (-3, \infty)$
2. $(1, \infty)$
3. $(-\infty, 5]$
4. $(1/2, \infty)$
5. $[1, \infty)$
6. $[-1, 1]$
7. $(-\infty, \infty)$
8. $(-\infty, 0) \cup (0, \infty)$
9. $(-1, 1)$
10. $(-\infty, \infty)$

Section 1.4 Exercises

1. $(f+g)(x) = 2x - 1 + x^2$; $(f-g)(x) = 2x - 1 - x^2$; $(fg)(x) = (2x-1)(x^2) = 2x^3 - x^2$.

There are no restrictions on any of the domains, so all three domains are $(-\infty, \infty)$.

2. $(f+g)(x) = (x-1)^2 + 3 - x = x^2 - 2x + 1 + 3 - x = x^2 - 3x + 4$;
$(f-g)(x) = (x-1)^2 - 3 + x = x^2 - 2x + 1 - 3 + x = x^2 - x - 2$;
$(fg)(x) = (x-1)^2(3-x) = (x^2 - 2x + 1)(3 - x) = 3x^2 - x^3 - 6x + 2x^2 + 3 - x = -x^3 + 5x^2 - 7x + 3$.

There are no restrictions on any of the domains, so all three domains are $(-\infty, \infty)$.

3. $(f+g)(x) = \sqrt{x} + \sin x$; $(f-g)(x) = \sqrt{x} - \sin x$; $(fg)(x) = \sqrt{x}\sin x$.
Domain in each case is $[0, \infty)$. For $\sqrt{x}$, $x \geq 0$. For $\sin x$, $-\infty < x < \infty$.

4. $(f+g)(x) = \sqrt{x+5} + |x+3|$;
$(f-g)(x) = \sqrt{x+5} - |x+3|$;
$(fg)(x) = \sqrt{x+5}\,|x+3|$.
All three expressions contain $\sqrt{x+5}$, so $x + 5 \geq 0$ and $x \geq -5$; all three domains are $[-5, \infty)$. For $|x+3|$, $-\infty < x < \infty$.

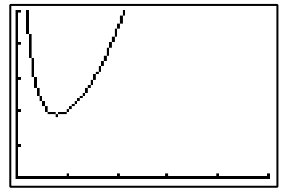
5. $(f/g)(x) = \dfrac{\sqrt{x+3}}{x^2}$; $x + 3 \geq 0$ and $x \neq 0$, so the domain is $[-3, 0) \cup (0, \infty)$.
$(g/f)(x) = \dfrac{x^2}{\sqrt{x+3}}$; $x + 3 > 0$, so the domain is $(-3, \infty)$.

6. $(f/g)(x) = \dfrac{\sqrt{x-2}}{\sqrt{x+4}} = \sqrt{\dfrac{x-2}{x+4}}$; $x - 2 \geq 0$ and $x + 4 > 0$, so $x \geq 2$ and $x > -4$; the domain is $[2, \infty)$.
$(g/f)(x) = \dfrac{\sqrt{x+4}}{\sqrt{x-2}} = \sqrt{\dfrac{x+4}{x-2}}$; $x + 4 \geq 0$ and $x - 2 > 0$, so $x \geq -4$ and $x > 2$; the domain is $(2, \infty)$.

7. $(f/g)(x) = \dfrac{x^2}{\sqrt{1-x^2}}$. The denominator cannot be zero and the term under the square root must be positive, so $1 - x^2 > 0$. Therefore, $x^2 < 1$, which means that $-1 < x < 1$. The domain is $(-1, 1)$.
$(g/f)(x) = \dfrac{\sqrt{1-x^2}}{x^2}$. The term under the square root must be nonnegative, so $1 - x^2 \geq 0$ (or $x^2 \leq 1$). The denominator cannot be zero, so $x \neq 0$. Therefore, $-1 \leq x < 0$ or $0 < x \leq 1$. The domain is $[-1, 0) \cup (0, 1]$.

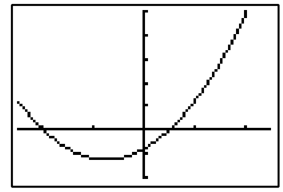
8. $(f/g)(x) = \dfrac{x^3}{\sqrt[3]{1-x^3}}$. The denominator cannot be 0, so $1 - x^3 \neq 0$ and $x^3 \neq 1$. This means that $x \neq 1$. There are no restrictions on x in the numerator. The domain is $(-\infty, 1) \cup (1, \infty)$.
$(g/f)(x) = \dfrac{\sqrt[3]{1-x^3}}{x^3}$. The denominator cannot be 0, so $x^3 \neq 0$ and $x \neq 0$. There are no restrictions on x in the numerator. The domain is $(-\infty, 0) \cup (0, \infty)$.

9.



$[0, 5]$ by $[0, 5]$

10.



$[-5, 5]$ by $[-10, 25]$

11. $(f \circ g)(3) = f(g(3)) = f(4) = 5$;
$(g \circ f)(-2) = g(f(-2)) = g(-7) = -6$

12. $(f \circ g)(3) = f(g(3)) = f(3) = 8$;
$(g \circ f)(-2) = g(f(-2)) = g(3) = 3$

13. $(f \circ g)(3) = f(g(3)) = f(\sqrt{3+1}) = f(2) = 2^2 + 4 = 8$;
$(g \circ f)(-2) = g(f(-2)) = g((-2)^2 + 4) = g(8) = \sqrt{8+1} = 3$

14. $(f \circ g)(3) = f(g(3)) = f(9 - 3^2) = f(0) = \dfrac{0}{0+1} = 0$;
$(g \circ f)(-2) = g(f(-2)) = g\left(\dfrac{-2}{-2+1}\right) = g(2) = 9 - 2^2 = 5$

15. $f(g(x)) = 3(x-1) + 2 = 3x - 3 + 2 = 3x - 1$. Because both f and g have domain $(-\infty, \infty)$, the domain of $f(g(x))$ is $(-\infty, \infty)$.
$g(f(x)) = (3x + 2) - 1 = 3x + 1$; again, the domain is $(-\infty, \infty)$.

16. $f(g(x)) = \left(\dfrac{1}{x-1}\right)^2 - 1 = \dfrac{1}{(x-1)^2} - 1$. The domain of g is $x \neq 1$, while the domain of f is $(-\infty, \infty)$, so the domain of $f(g(x))$ is $x \neq 1$, or $(-\infty, 1) \cup (1, \infty)$.
$g(f(x)) = \dfrac{1}{(x^2 - 1) - 1} = \dfrac{1}{x^2 - 2}$.
The domain of f is $(-\infty, \infty)$, while the domain of g is $(-\infty, 1) \cup (1, \infty)$, so $g(f(x))$ requires that $f(x) \neq 1$. This means $x^2 - 1 \neq 1$, or $x^2 \neq 2$, so the domain of $g(f(x))$ is $x \neq \pm\sqrt{2}$, or $(-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty)$.

17. $f(g(x)) = (\sqrt{x+1})^2 - 2 = x + 1 - 2 = x - 1$. The domain of g is $x \geq -1$, while the domain of f is $(-\infty, \infty)$, so the domain of $f(g(x))$ is $x \geq -1$, or $[-1, \infty)$.
$g(f(x)) = \sqrt{(x^2 - 2) + 1} = \sqrt{x^2 - 1}$. The domain of f is $(-\infty, \infty)$, while the domain of g is $[-1, \infty)$, so $g(f(x))$ requires that $f(x) \geq -1$.
This means $x^2 - 2 \geq -1$, or $x^2 \geq 1$, which means $x \leq -1$ or $x \geq 1$. Therefore the domain of $g(f(x))$ is $(-\infty, -1] \cup [1, \infty)$.

18. $f(g(x)) = \dfrac{1}{\sqrt{x} - 1}$. The domain of g is $x \geq 0$, while the domain of f is $(-\infty, 1) \cup (1, \infty)$, so $f(g(x))$ requires that $x \geq 0$ and $g(x) \neq 1$, or $x \geq 0$, and $x \neq 1$. The domain of $f(g(x))$ is $[0, 1) \cup (1, \infty)$.
$g(f(x)) = \sqrt{\dfrac{1}{x-1}} = \dfrac{1}{\sqrt{x-1}}$. The domain of f is $x \neq 1$, while the domain of g is $[0, \infty)$, so $g(f(x))$ requires that $x \neq 1$ and $f(x) \geq 0$, or $x \neq 1$ and $\dfrac{1}{x-1} \geq 0$. The latter occurs if $x - 1 > 0$, so the domain of $g(f(x))$ is $(1, \infty)$.

19. $f(g(x)) = f(\sqrt{1 - x^2}) = (\sqrt{1 - x^2})^2 = 1 - x^2$; the domain is $[-1, 1]$.
$g(f(x)) = g(x^2) = \sqrt{1 - (x^2)^2} = \sqrt{1 - x^4}$; the domain is $[-1, 1]$.

20. $f(g(x)) = f(\sqrt[3]{1 - x^3}) = (\sqrt[3]{1 - x^3})^3 = 1 - x^3$; the domain is $(-\infty, \infty)$.
$g(f(x)) = g(x^3) = \sqrt[3]{1 - (x^3)^3} = \sqrt[3]{1 - x^9}$; the domain is $(-\infty, \infty)$.

21. $f(g(x)) = f\left(\frac{1}{3x}\right) = \frac{1}{2(1/3x)} = \frac{1}{2/3x} = \frac{3x}{2}$; the domain is $(-\infty, 0) \cup (0, \infty)$.
$g(f(x)) = g\left(\frac{1}{2x}\right) = \frac{1}{3(1/2x)} = \frac{1}{3/2x} = \frac{2x}{3}$; the domain is $(-\infty, 0) \cup (0, \infty)$.

22. $f(g(x)) = f\left(\frac{1}{x-1}\right) = \frac{1}{(1/(x-1)) + 1} = \frac{1}{(1 + (x-1))/(x-1)} = \frac{1}{x/(x-1)} = \frac{x-1}{x}$; the domain is all reals except 0 and 1.
$g(f(x)) = g\left(\frac{1}{x+1}\right) = \frac{1}{(1/(x+1)) - 1} = \frac{1}{(1 + (x-1))/(x+1)} = \frac{1}{x/(x+1)} = \frac{x+1}{x}$; the domain is all reals except 0 and 1.

23. One possibility: $f(x) = \sqrt{x}$ and $g(x) = x^2 - 5x$

24. One possibility: $f(x) = (x + 1)^2$ and $g(x) = x^3$

25. One possibility: $f(x) = |x|$ and $g(x) = 3x - 2$

26. One possibility: $f(x) = 1/x$ and $g(x) = x^3 - 5x + 3$

27. One possibility: $f(x) = x^5 - 2$ and $g(x) = x - 3$

28. One possibility: $f(x) = e^x$ and $g(x) = \sin x$

29. One possibility: $f(x) = \cos x$ and $g(x) = \sqrt{x}$.

30. One possibility: $f(x) = x^2 + 1$ and $g(x) = \tan x$.

31. $r = 48 + 0.03t$ in., so $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(48 + 0.03t)^3$; when $t = 300$,
$V = \frac{4}{3}\pi(48 + 9)^3 = 246{,}924\pi \approx 775{,}734.6 \text{ in}^3$.

32. The original diameter of each snowball is 4 in., so the original radius is 2 in. and the original volume $V = \frac{4}{3}\pi r^3 \approx 33.5 \text{ in}^3$. The new volume is $V = 33.5 - t$, where t is the number of 40-day periods. At the end of 360 days, the new volume is $V = 33.5 - 9 = 24.5$. Since $V = \frac{4}{3}\pi r^3$, we know that $r = \sqrt[3]{\frac{3V}{4\pi}} \approx 1.8$ in. The diameter, then, is 2 times r, or ≈ 3.6 in.

33. The initial area is $(5)(7) = 35 \text{ km}^2$. The new length and width are $l = 5 + 2t$ and $w = 7 + 2t$, so $A = lw = (5 + 2t)(7 + 2t)$. Solve $(7 + 2t)(5 + 2t) = 175$ (5 times its original size), either graphically or algebraically: the positive solution is $t \approx 3.63$ seconds.

34. The initial volume is $(5)(7)(3) = 105 \text{ cm}^3$. The new length, width, and height are $l = 5 + 2t$, $w = 7 + 2t$, and $h = 3 + 2t$, so the new volume is $V = (5 + 2t)(7 + 2t)(3 + 2t)$. Solve graphically $(5 + 2t)(7 + 2t)(3 + 2t) \geq 525$ (5 times the original volume): $t \approx 1.62$ sec.

35. $3(1) + 4(1) = 3 + 4 = 7 \neq 5$
$3(4) + 4(-2) = 12 - 8 = 4 \neq 5$
$3(3) + 4(-1) = 9 - 4 = 5$
The answer is $(3, -1)$.

36. $(5)^2 + (1)^2 = 25 + 1 = 26 \neq 25$
$(3)^2 + (4)^2 = 9 + 16 = 25$
$(0)^2 + (-5)^2 = 0 + 25 = 25$
The answer is $(3, 4)$ and $(0, -5)$.

37. $y^2 = 25 - x^2$, $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$

38. $y^2 = 25 - x$, $y = \sqrt{25 - x}$ and $y = -\sqrt{25 - x}$

39. $y^2 = x^2 - 25$, $y = \sqrt{x^2 - 25}$ and $y = -\sqrt{x^2 - 25}$

40. $y^2 = 3x^2 - 25$, $y = \sqrt{3x^2 - 25}$ and $y = -\sqrt{3x^2 - 25}$

41. $x + |y| = 1 \Rightarrow |y| = -x + 1 \Rightarrow y = -x + 1$ or $y = -(-x + 1)$. $y = 1 - x$ and $y = x - 1$

42. $x - |y| = 1 \Rightarrow |y| = x - 1 \Rightarrow y = x - 1$ or $y = -(x - 1) = -x + 1$. $y = x - 1$ and $y = 1 - x$

43. $y^2 = x^2 \Rightarrow y = x$ and $y = -x$ or $y = |x|$ and $y = -|x|$

44. $y^2 = x \Rightarrow y = \sqrt{x}$ and $y = -\sqrt{x}$

45. False. If $g(x) = 0$, then $\left(\frac{f}{g}\right)(x)$ is not defined and 0 is not in the domain of $\left(\frac{f}{g}\right)(x)$, even though 0 may be in the domains of both $f(x)$ and $g(x)$.

46. False. For a number to be in the domain of $(fg)(x)$, it must be in the domains of *both* $f(x)$ and $g(x)$, so that $f(x)$ and $g(x)$ are both defined.

47. Composition of functions isn't necessarily commutative. The answer is C.

48. $g(x) = \sqrt{4 - x}$ cannot equal zero and the term under the square root must be positive, so x can be any real number less than 4. The answer is A.

49. $(f \circ f)(x) = f(x^2 + 1) = (x^2 + 1)^2 + 1 = (x^4 + 2x^2 + 1) + 1 = x^4 + 2x^2 + 2$. The answer is E.

50. $y = |x| \Rightarrow y = x, y = -x; y = -x \Rightarrow x = -y; x = -y$ or $x = y \Rightarrow x^2 = y^2$. The answer is B.

51. If $f(x) = e^x$ and $g(x) = 2 \ln x$, then $f(g(x)) = f(2 \ln x) = e^{2 \ln x} = (e^{\ln x})^2 = x^2$. The domain is $(0, \infty)$.
If $f(x) = (x^2 + 2)^2$ and $g(x) = \sqrt{x - 2}$, then $f(g(x)) = f(\sqrt{x - 2}) = ((\sqrt{x - 2})^2 + 2)^2 = (x - 2 + 2)^2 = x^2$. The domain is $[2, \infty)$.
If $f(x) = (x^2 - 2)^2$ and $g(x) = \sqrt{2 - x}$, then $f(g(x)) = f(\sqrt{2 - x}) = ((\sqrt{2 - x})^2 - 2)^2 = (2 - x - 2)^2 = x^2$. The domain is $(-\infty, 2]$.

If $f(x) = \dfrac{1}{(x-1)^2}$ and $g(x) = \dfrac{x+1}{x}$, then

$$f(g(x)) = f\left(\frac{x+1}{x}\right) = \frac{1}{\left(\frac{x+1}{x} - 1\right)^2} =$$

$$\frac{1}{\left(\frac{x+1-x}{x}\right)^2} = \frac{1}{\frac{1}{x^2}} = x^2.$$ The domain is $x \neq 0$.

If $f(x) = x^2 - 2x + 1$ and $g(x) = x + 1$, then $f(g(x)) = f(x+1) = (x+1)^2 - 2(x+1) + 1 = ((x+1) - 1)^2 = x^2$. The domain is $(-\infty, \infty)$.

If $f(x) = \left(\dfrac{x+1}{x}\right)^2$ and $g(x) = \dfrac{1}{x-1}$, then

$$f(g(x)) = f\left(\frac{1}{x-1}\right) = \left(\frac{\frac{1}{x-1} + 1}{\frac{1}{x-1}}\right)^2 =$$

$$\left(\frac{\frac{1+x-1}{x-1}}{\frac{1}{x-1}}\right)^2 = x^2.$$ The domain is $x \neq 1$.

f	g	D
e^x	$2 \ln x$	$(0, \infty)$
$(x^2+2)^2$	$\sqrt{x-2}$	$[2, \infty)$
$(x^2-2)^2$	$\sqrt{2-x}$	$(-\infty, 2]$
$\dfrac{1}{(x-1)^2}$	$\dfrac{x+1}{x}$	$x \neq 0$
$x^2 - 2x + 1$	$x + 1$	$(-\infty, \infty)$
$\left(\dfrac{x+1}{x}\right)^2$	$\dfrac{1}{x-1}$	$x \neq 1$

52. (a) $(fg)(x) = x^4 - 1 = (x^2+1)(x^2-1) = f(x) \cdot (x^2 - 1)$, so $g(x) = x^2 - 1$.

(b) $(f+g)(x) = 3x^2 \Rightarrow 3x^2 - (x^2+1) = 2x^2 - 1 = g(x)$.

(c) $(f/g)(x) = 1 \Rightarrow f(x) = g(x)$. So $g(x) = x^2 + 1$.

(d) $f(g(x)) = 9x^4 + 1$ and $f(x) = x^2 + 1$. If $g(x) = 3x^2$, then $f(g(x)) = f(3x^2) = (3x^2)^2 + 1 = 9x^4 + 1$.

(e) $g(f(x)) = 9x^4 + 1$ and $f(x) = x^2 + 1$. Then $g(x^2+1) = 9x^4 + 1 = 9((x^2+1) - 1)^2 + 1$, so $g(x) = 9(x-1)^2 + 1$.

53. (a) $(f+g)(x) = (g+f)(x) = f(x)$ if $g(x) = 0$.

(b) $(fg)(x) = (gf)(x) = f(x)$ if $g(x) = 1$.

(c) $(f \circ g)(x) = (g \circ f)(x) = f(x)$ if $g(x) = x$.

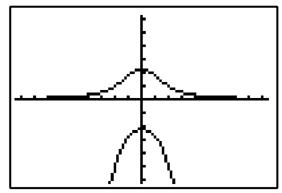
54. Yes, by definition, function composition is associative. That is, $(f \circ (g \circ h))(x) = f(g(h))(x)$ and $((f \circ g) \circ h)(x) = f(g(h))(x)$.

55. $y^2 + x^2 y - 5 = 0$. Using the quadratic formula,

$$y = \frac{-x^2 \pm \sqrt{(x^2)^2 - 4(1)(-5)}}{2} = \frac{-x^2 \pm \sqrt{x^4 + 20}}{2}$$

so, $y_1 = \dfrac{-x^2 + \sqrt{x^4+20}}{2}$

and $y_2 = \dfrac{-x^2 - \sqrt{x^4+20}}{2}$.



$[-9.4, 9.4]$ by $[-6.2, 6.2]$