■ Section 1.5 Parametric Relations and Inverses

Exploration 1

1. T starts at −4, at the point (8, −3). It stops at T = 2, at the point (8, 3). 61 points are computed.
2. The graph is smoother because the plotted points are closer together.
3. The graph is less smooth because the plotted points are further apart. In CONNECT mode, they are connected by straight lines.
4. The smaller the Tstep, the slower the graphing proceeds. This is because the calculator has to compute more X and Y values.
5. The grapher skips directly from the point (0, −1) to the point (0, 1), corresponding to the T-values T = −2 and T = 0. The two points are connected by a straight line, hidden by the Y-axis.
6. With the Tmin set at −1, the grapher begins at the point (−1, 0), missing the bottom of the curve entirely.
7. Leave everything else the same, but change Tmin back to −4 and Tmax to −1.

Quick Review 1.5

1. $3y = x + 6$, so $y = \dfrac{x+6}{3} = \dfrac{1}{3}x + 2$
2. $0.5y = x - 1$, so $y = \dfrac{x-1}{0.5} = 2x - 2$
3. $y^2 = x - 4$, so $y = \pm\sqrt{x-4}$
4. $y^2 = x + 6$, so $y = \pm\sqrt{x+6}$
5. $$\begin{aligned} x(y+3) &= y - 2 \\ xy + 3x &= y - 2 \\ xy - y &= -3x - 2 \\ y(x-1) &= -(3x+2) \\ y &= -\frac{3x+2}{x-1} = \frac{3x+2}{1-x} \end{aligned}$$

6. $$\begin{aligned} x(y+2) &= 3y-1 \\ xy+2x &= 3y-1 \\ xy-3y &= -2x-1 \\ y(x-3) &= -(2x+1) \\ y &= -\frac{2x+1}{x-3} = \frac{2x+1}{3-x} \end{aligned}$$

7. $$\begin{aligned} x(y-4) &= 2y+1 \\ xy-4x &= 2y+1 \\ xy-2y &= 4x+1 \\ y(x-2) &= 4x+1 \\ y &= \frac{4x+1}{x-2} \end{aligned}$$

8. $$\begin{aligned} x(3y-1) &= 4y+3 \\ 3xy-x &= 4y+3 \\ 3xy-4y &= x+3 \\ y(3x-4) &= x+3 \\ y &= \frac{x+3}{3x-4} \end{aligned}$$

9. $x = \sqrt{y+3}$, $y \ge -3$ [and $x \ge 0$]
$x^2 = y+3$, $y \ge -3$, and $x \ge 0$
$y = x^2 - 3$, $y \ge -3$, and $x \ge 0$

10. $x = \sqrt{y-2}$, $y \ge 2$ [and $x \ge 0$]
$x^2 = y-2$, $y \ge 2$, and $x \ge 0$
$y = x^2 + 2$, $y \ge 2$, and $x \ge 0$

Section 1.5 Exercises

1. $x = 3(2) = 6$, $y = 2^2 + 5 = 9$. The answer is $(6, 9)$.

2. $x = 5(-2) - 7 = -17$, $y = 17 - 3(-2) = 23$. The answer is $(-17, 23)$.

3. $x = 3^3 - 4(3) = 15$, $y = \sqrt{3+1} = 2$. The answer is $(15, 2)$.

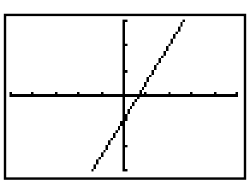
4. $x = |-8+3| = 5$, $y = \frac{1}{-8} = -\frac{1}{8}$

5. (a)

t	$(x, y) = (2t, 3t-1)$
-3	$(-6, -10)$
-2	$(-4, -7)$
-1	$(-2, -4)$
0	$(0, -1)$
1	$(2, 2)$
2	$(4, 5)$
3	$(6, 8)$

(b) $t = \frac{x}{2}$, $y = 3\left(\frac{x}{2}\right) - 1 = 1.5x - 1$. This is a function.

(c)



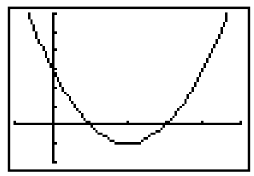
[–5, 5] by [–3, 3]

6. (a)

t	$(x, y) = (t+1, t^2-2t)$
-3	$(-2, 15)$
-2	$(-1, 8)$
-1	$(0, 3)$
0	$(1, 0)$
1	$(2, -1)$
2	$(3, 0)$
3	$(4, 3)$

(b) $t = x - 1$, $y = (x-1)^2 - 2(x-1)$
$= x^2 - 2x + 1 - 2x + 2$
$= x^2 - 4x + 3$
This is a function.

(c)



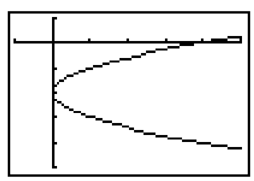
[–1, 5] by [–2, 6]

7. (a)

t	$(x, y) = (t^2, t-2)$
-3	$(9, -5)$
-2	$(4, -4)$
-1	$(1, -3)$
0	$(0, -2)$
1	$(1, -1)$
2	$(4, 0)$
3	$(9, 1)$

(b) $t = y + 2$, $x = (y+2)^2$. This is not a function.

(c)



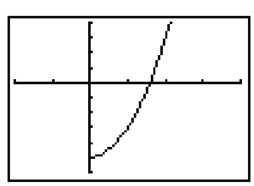
[–1, 5] by [–5, 1]

8. (a)

t	$(x, y) = (\sqrt{t}, 2t-5)$
-3	$\sqrt{-3}$ not defined
-2	$\sqrt{-2}$ not defined
-1	$\sqrt{-1}$ not defined
0	$(0, -5)$
1	$(1, -3)$
2	$(\sqrt{2}, -1)$
3	$(\sqrt{3}, 1)$

(b) $t = x^2$, $y = 2x^2 - 5$. This is a function.

(c)



[–2, 4] by [–6, 4]

9. (a) By the vertical line test, the relation is not a function.

(b) By the horizontal line test, the relation's inverse is a function.

10. (a) By the vertical line test, the relation is a function.

(b) By the horizontal line test, the relation's inverse is not a function.

11. (a) By the vertical line test, the relation is a function.

(b) By the horizontal line test, the relation's inverse is a function.

12. (a) By the vertical line test, the relation is not a function.

(b) By the horizontal line test, the relation's inverse is a function.

13. $y = 3x - 6 \Rightarrow \quad x = 3y - 6$
$$3y = x + 6$$
$$f^{-1}(x) = y = \frac{x+6}{3} = \frac{1}{3}x + 2; (-\infty, \infty)$$

14. $y = 2x + 5 \Rightarrow \quad x = 2y + 5$
$$2y = x - 5$$
$$f^{-1}(x) = y = \frac{x-5}{2} = \frac{1}{2}x - \frac{5}{2};$$
$(-\infty, \infty)$

15. $y = \dfrac{2x-3}{x+1} \Rightarrow \quad x = \dfrac{2y-3}{y+1}$
$$x(y+1) = 2y - 3$$
$$xy + x = 2y - 3$$
$$xy - 2y = -x - 3$$
$$y(x-2) = -(x+3)$$
$$f^{-1}(x) = y = -\frac{x+3}{x-2} = \frac{x+3}{2-x};$$
$(-\infty, 2) \cup (2, \infty)$

16. $y = \dfrac{x+3}{x-2} \Rightarrow \quad x = \dfrac{y+3}{y-2}$
$$x(y-2) = y + 3$$
$$xy - 2x = y + 3$$
$$xy - y = 2x + 3$$
$$y(x-1) = 2x + 3$$
$$f^{-1}(x) = y = \frac{2x+3}{x-1};$$
$x \neq 1$ or $(-\infty, 1) \cup (1, \infty)$

17. $y = \sqrt{x-3}, x \geq 3, y \geq 0 \Rightarrow$
$$x = \sqrt{y-3}, \quad x \geq 0, y \geq 3$$
$$x^2 = y - 3, \quad x \geq 0, y \geq 3$$
$$f^{-1}(x) = y = x^2 + 3, \quad x \geq 0$$

18. $y = \sqrt{x+2}, x \geq -2, y \geq 0 \Rightarrow$
$$x = \sqrt{y+2}, \quad x \geq 0, y \geq -2$$
$$x^2 = y + 2, \quad x \geq 0, y \geq -2$$
$$f^{-1}(x) = y = x^2 - 2, \quad x \geq 0$$

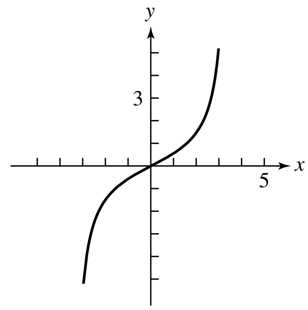
19. $y = x^3 \Rightarrow \quad x = y^3$
$$f^{-1}(x) = y = \sqrt[3]{x}; (-\infty, \infty)$$

20. $y = x^3 + 5 \Rightarrow \quad x = y^3 + 5$
$$x - 5 = y^3$$
$$f^{-1}(x) = y = \sqrt[3]{x-5}; (-\infty, \infty)$$

21. $y = \sqrt[3]{x+5} \Rightarrow \quad x = \sqrt[3]{y+5}$
$$x^3 = y + 5$$
$$f^{-1}(x) = y = x^3 - 5; (-\infty, \infty)$$

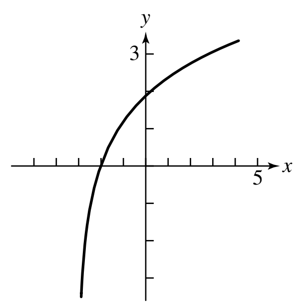
22. $y = \sqrt[3]{x-2} \Rightarrow \quad x = \sqrt[3]{y-2}$
$$x^3 = y - 2$$
$$f^{-1}(x) = y = x^3 + 2; (-\infty, \infty)$$

23. One-to-one



24. Not one-to-one

25. One-to-one



26. Not one-to-one

27. $f(g(x)) = 3\left[\frac{1}{3}(x+2)\right] - 2 = x + 2 - 2 = x;$
$$g(f(x)) = \frac{1}{3}[(3x-2)+2] = \frac{1}{3}(3x) = x$$

28. $f(g(x)) = \frac{1}{4}[(4x-3)+3] = \frac{1}{4}(4x) = x;$
$$g(f(x)) = 4\left[\frac{1}{4}(x+3)\right] - 3 = x + 3 - 3 = x$$

29. $f(g(x)) = [(x-1)^{1/3}]^3 + 1 = (x-1)^1 + 1$
$$= x - 1 + 1 = x;$$
$$g(f(x)) = [(x^3+1)-1]^{1/3} = (x^3)^{1/3} = x^1 = x$$

30. $f(g(x)) = \dfrac{7}{\frac{7}{x}} = \dfrac{7}{1} \cdot \dfrac{x}{7} = x; g(f(x)) = \dfrac{7}{\frac{7}{x}} = \dfrac{7}{1} \cdot \dfrac{x}{7} = x$

31. $f(g(x)) = \dfrac{\frac{1}{x-1} + 1}{\frac{1}{x-1}} = (x-1)\left(\dfrac{1}{x-1} + 1\right)$
$$= 1 + x - 1 = x;$$
$$g(f(x)) = \frac{1}{\frac{x+1}{x} - 1} = \left(\frac{1}{\frac{x+1}{x} - 1}\right) \cdot \frac{x}{x}$$
$$= \frac{x}{x+1-x} = \frac{x}{1} = x$$

32. $f(g(x)) = \dfrac{\dfrac{2x+3}{x-1}+3}{\dfrac{2x+3}{x-1}-2} = \left(\dfrac{\dfrac{2x+3}{x-1}+3}{\dfrac{2x+3}{x-1}-2}\right)\cdot\left(\dfrac{x-1}{x-1}\right)$

$= \dfrac{2x+3+3(x-1)}{2x+3-2(x-1)} = \dfrac{5x}{5} = x;$

$g(f(x)) = \dfrac{2\left(\dfrac{x+3}{x-2}\right)+3}{\dfrac{x+3}{x-2}-1}$

$= \left[\dfrac{2\left(\dfrac{x+3}{x-2}\right)+3}{\dfrac{x+3}{x-2}-1}\right]\cdot\left(\dfrac{x-2}{x-2}\right)$

$= \dfrac{2(x+3)+3(x-2)}{x+3-(x-2)} = \dfrac{5x}{5} = x$

33. **(a)** $y = (1.08)(100) = 108$ euros

(b) $x = \dfrac{y}{1.08} = \dfrac{25}{27}y$. This converts euros ($x$) to dollars ($y$).

(c) $x = (0.9259)(48) = \$44.44$

34. **(a)** $9c(x) = 5(x-32)$

$\dfrac{9}{5}c(x) = x - 32$

$\dfrac{9}{5}c(x) + 32 = x$

In this case, $c(x)$ becomes x, and x becomes $c^{-1}(x)$ for the inverse. So, $c^{-1}(x) = \dfrac{9}{5}x + 32$. This converts Celsius temperature to Fahrenheit temperature.

(b) $(k \circ c)(x) = k(c(x)) = k\left(\dfrac{5}{9}(x-32)\right)$

$\dfrac{5}{9}(x-32) + 273.16 = \dfrac{5}{9}x + 255.38$. This is used to convert Fahrenheit temperature to Kelvin temperature.

35. $y = e^x$ and $y = \ln x$ are inverses. If we restrict the domain of the function $y = x^2$ to the interval $[0, \infty)$, then the restricted function and $y = \sqrt{x}$ are inverses.

36. $y = x$ and $y = 1/x$ are their own inverses.

37. $y = |x|$

38. $y = x$

39. True. All the ordered pairs swap domain and range values.

40. True. This is a parametrization of the line $y = 2x + 1$.

41. The inverse of the relation given by $x^2y + 5y = 9$ is the relation given by $y^2x + 5x = 9$.

$(1)^2(2) + 5(2) = 2 + 10 = 12 \neq 9$
$(1)^2(-2) + 5(-2) = -2 - 10 = -12 \neq 9$
$(2)^2(-1) + 5(-1) = -4 - 5 = -9 \neq 9$
$(-1)^2(2) + 5(2) = 2 + 10 = 12 \neq 9$
$(-2)^2(1) + 5(1) = 4 + 5 = 9$

The answer is E.

42. The inverse of the relation given by $xy^2 - 3x = 12$ is the relation given by $yx^2 - 3y = 12$.

$(-4)(0)^2 - 3(-4) = 0 + 12 = 12$
$(1)(4)^2 - 3(1) = 16 - 3 = 13 \neq 12$
$(2)(3)^2 - 3(2) = 18 - 6 = 12$
$(12)(2)^2 - 3(12) = 48 - 36 = 12$
$(-6)(1)^2 - 3(-6) = -6 + 18 = 12$

The answer is B.

43. $f(x) = 3x - 2$

$y = 3x - 2$

The inverse relation is

$x = 3y - 2$

$x + 2 = 3y$

$\dfrac{x+2}{3} = y$

$f^{-1}(x) = \dfrac{x+2}{3}$

The answer is C.

44. $f(x) = x^3 + 1$

$y = x^3 + 1$

The inverse relation is

$x = y^3 + 1$

$x - 1 = y^3$

$\sqrt[3]{x-1} = y$

$f^{-1}(x) = \sqrt[3]{x-1}$

The answer is A.

45. (Answers may vary.)

(a) If the graph of f is unbroken, its reflection in the line $y = x$ will be also.

(b) Both f and its inverse must be one-to-one in order to be inverse functions.

(c) Since f is odd, $(-x, -y)$ is on the graph whenever (x, y) is. This implies that $(-y, -x)$ is on the graph of f^{-1} whenever (x, y) is. That implies that f^{-1} is odd.

(d) Let $y = f(x)$. Since the ratio of Δy to Δx is positive, the ratio of Δx to Δy is positive. Any ratio of Δy to Δx on the graph of f^{-1} is the same as some ratio of Δx to Δy on the graph of f, hence positive. This implies that f^{-1} is increasing.

46. (Answers may vary.)

(a) $f(x) = e^x$ has a horizontal asymptote; $f^{-1}(x) = \ln x$ does not.

(b) $f(x) = e^x$ has domain all real numbers; $f^{-1}(x) = \ln x$ does not.

(c) $f(x) = e^x$ has a graph that is bounded below; $f^{-1}(x) = \ln x$ does not.

(d) $f(x) = \dfrac{x^2 - 25}{x - 5}$ has a removable discontinuity at $x = 5$ because its graph is the line $y = x + 5$ with the point $(5, 10)$ removed. The inverse function is the line $y = x - 5$ with the point $(10, 5)$ removed. This function has a removable discontinuity, but not at $x = 5$.

47. (a) $\frac{\Delta y}{\Delta x} = \frac{97 - 70}{88 - 52} = \frac{27}{36} = 0.75$, which gives us the slope of the equation. To find the rest of the equation, we use one of the initial points

$$y - 70 = 0.75(x - 52)$$
$$y = 0.75x - 39 + 70$$
$$y = 0.75x + 31$$

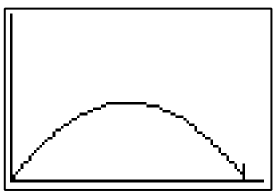
(b) To find the inverse, we substitute y for x and x for y, and then solve for y:

$$x = 0.75y + 31$$
$$x - 31 = 0.75y$$
$$y = \frac{4}{3}(x - 31)$$

The inverse function converts scaled scores to raw scores.

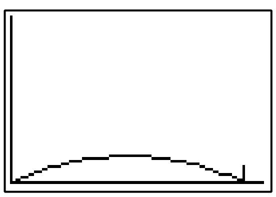
48. The function must be increasing so that the *order* of the students' grades, top to bottom, will remain the same after scaling as it is before scaling. A student with a raw score of 136 gets dropped to 133, but that will still be higher than the scaled score for a student with 134.

49. (a) It does not clear the fence.



[0, 350] by [0, 300]

(b) It still does not clear the fence.



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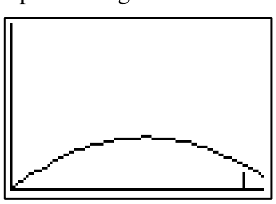
(c) Optimal angle is 45°. It clears the fence.



[0, 350] by [0, 300]

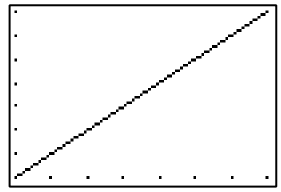
50. (a)

$$x = \left(\frac{3^{1.7}}{30}(y - 65)\right)^{\frac{1}{1.7}} + 1$$
$$x - 1 = \left(\frac{3^{1.7}}{30}(y - 65)\right)^{\frac{1}{1.7}}$$
$$(x - 1)^{1.7} = \left(\frac{3^{1.7}}{30}(y - 65)\right)$$
$$\frac{30}{3^{1.7}}(x - 1)^{1.7} = y - 65$$
$$y = \frac{30}{3^{1.7}}(x - 1)^{1.7} + 65$$

This can be use to convert GPA's to percentage grades.

(b) Yes; x is restricted to the domain [1, 4.28].

(c)



[65, 100] by [65, 100]

The composition function of $(y \circ y^{-1})(x)$is $y = x$, so they are inverses.

51. When $k = 1$, the scaling function is linear. Opinions will vary as to which is the best value of k.