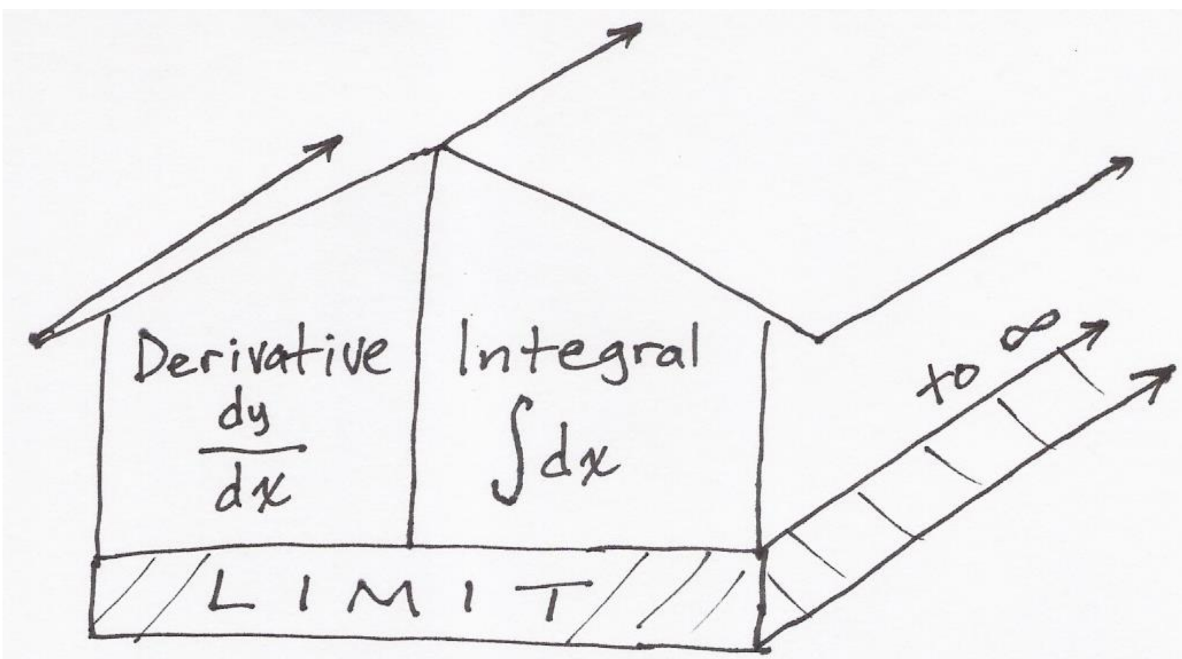
§1.1—Limits & Continuity

What do you see below?

We are building the "House of Calculus," one side at a time . . . and we need a solid FOUNDATION.



Example 1:

(Calculator) For $f(x)=x^2$

(a) fill in the following chart

x	2.9	2.99	2.999	3.1	3.01	3.001
$f(x)$						

(b) What do these values tell us about f in the **<u>neighborhood</u>** of $x=3$?

(c) Based on the chart above, what do you think is the value of $f(3)$?

(d) What IS $f(3)$?

Example 2:

(Calculator) If $g(x)=\dfrac{x^3-3x^2}{x-3}$,

(a) fill in the following chart

x	2.9	2.99	2.999	3.1	3.01	3.001
$g(x)$						

(b) What do these values tell us about g in the **<u>neighborhood</u>** of $x=3$?

(c) Based on the chart above, what do you think is the value of $g(3)$?

(d) What IS $g(3)$?

(e) Is there a still a way to mathematically communicate the result from the chart above?

Going where we can't go . . .

This interruption to the flow of the graph of g in Example 2 is called a **removable point discontinuity**, or a **hole** in the graph of g.

Simply evaluating a function at a particular value is insufficient for understanding the behavior of some types of functions at that point, especially functions with discontinuities at those points. There is, therefore, a need to come up with another method that will circumvent the possibility of going directly **to** a location, but rather **approaching** that location from either side of it. This is the limit, and it has its own notation as you will see

The Limit is a Notion of Motion

Example 3:
Suppose you wanted to safely and smoooooothly cross a chasm in your car. <u>What three things would you need?</u> Would you be able to do so in the situation depicted in the photo to the right? What is missing?

If you were to build the bridge, would you know exactly where to build it to make crossing the chasm in a car a continuously smooth undertaking?

As you can see, the existence or non-existence of the **<u>bridge (function value)</u>** is independent of the existence of the **<u>roads leading in from either side (the limit from the left and right)</u>**. This is exactly why the limit is so important—it gives us a way to talk about the activity in the neighborhood of a point whether or not the graph (bridge) exists at that point or not.

Example 4:

For $g(x)=\dfrac{x^3-3x^2}{x-3}$, algebraically determine the coordinate of $\left(x, g\left(x\right)\right)$ the removable point discontinuity, then use limit notation to describe what is happening as the discontinuity is approached from either side.

<u>Theorem</u>:

$$\lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x) \Leftrightarrow \lim_{x\to c} f(x) = L$$

The theorem above essentially says that the limit will exist *if and only if* the two roads coming in from either side are aligning with each other, irrespective of whether there is a bridge connecting the two roads. **The limit describes the *y*-values to which the roads are leading! The function value pertains to the bridge itself at a single point.**

Example 5:

For $f(x)=\dfrac{x^2+4x+3}{x^2-3x-4}$, algebraically determine the following:

(a) $f\left(-1\right)$ (b) $\lim\limits_{x\to -1^-} f\left(x\right)$ (c) $\lim\limits_{x\to -1^+} f\left(x\right)$ (d) $\lim\limits_{x\to -1} f\left(x\right)$

If our goal is to safely and smoothly drive across the chasm in our car, what relationship among the road leading in, the bridge, and the road leading out must there be?

With the limit notation, we now have a way to define continuity at a point.

Continuity at a point (HUGELY IMPORTANT)

A function $f(x)$ is **continuous at a point** $x=c$ if

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

that is

road in = bridge = road out

Example 6:

Using the definition of continuity, determine whether the graph of $h(x)=\dfrac{x^3-4x}{x^2+8x-20}$ is continuous at the following. Justify.

(a) $x=2$

(b) $x=0$

$x=-10$

Removable Point Discontinuity

If for some function $f(x)$ at $x = c$,

$$\lim_{x \to c} f(x) = L \text{ but either } f(c) \text{ is undefined or } f(c) \neq L$$

$$\Leftrightarrow$$

$f(x)$ has a removable point discontinuity (hole) at (c, L)

We know that the function value and the limit value exist independently of one another, but the one-sided limits may also exist independent of each other.

Example 7:

For $f(x) = \begin{cases} x^2, & x < 2 \\ 2x - 1, & x > 2 \end{cases}$, algebraically determine the following:

(a) $f(2)$

(b) $\lim_{x \to 2^-} f(x)$

(c) $\lim_{x \to 2^+} f(x)$

(d) $\lim_{x \to 2} f(x)$

(e) continuity of *f* at $x = 2$. Justify.

Non-Removable Jump Discontinuity

If for some function $f(x)$ at $x = c$,

$\lim_{x \to c^-} f(x) = L_1$ and $\lim_{x \to c^+} f(x) = L_2$ where $L_1 \neq L_2$ (regardless of $f(c)$)

$$\Leftrightarrow$$

$f(x)$ has a non-removable jump discontinuity at $x = c$

Example 8:
Using the definition of continuity at a point, discuss the continuity of the following function. Justify.

$$f(x)=\begin{cases} x+2, & x<-2 \\ \dfrac{x^2}{2}, & -2\le x\le 2 \\ 2, & x>2 \end{cases}$$

The 3-step definition of continuity at a point can also provide us with a system of equations needed to find unknowns.

Example 9:
If $f(x)=\begin{cases} ax^2-b, & x<-1 \\ 4, & x=-1 \\ 2ax+b, & x>-1 \end{cases}$, find the values of *a* and *b* such that $f(x)$ is continuous at $x=-1$.

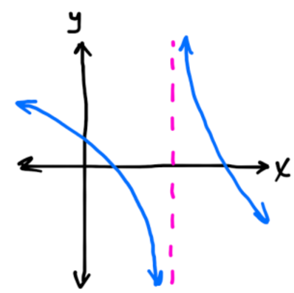
Before we move on to explore other discontinuities, it's worth noting something about particular functions that we KNOW to be continuous (such as polynomials, sine, cosine, exponential, etc.)

If a function is continuous at a point, then the function value and the limit value are the same at that point!

Example 10:
Evaluate each of the following limits:

(a) $\lim_{x\to 1}\left(4x^5-7x^4+3x^3-x^2+x-5\right)$ (b) $\lim_{x\to \frac{5\pi}{3}} \cos x$ (c) $\lim_{x\to e}\frac{\ln x}{2}$

In **Example 6**, we saw another type of discontinuity—a vertical asymptote (VA), officially called a **<u>Non-Removable Infinite Discontinuity</u>**.



Similar to a jump discontinuity, the limit will always fail to exist at a VA, but for a very different reason. As we approach a vertical asymptote from either side, there are only two options—**go down forever to negative infinity or go up forever to infinity . . . neither of which is a limit!**

Non-Removable Infinite Discontinuity

If for some function $f(x)$ at $x=c$,

$$\lim_{x\to c^+} f(x)=\infty \text{ or } \lim_{x\to c^+} f(x)=-\infty \text{ or } \lim_{x\to c^-} f(x)=\infty \text{ or } \lim_{x\to c^-} f(x)=-\infty$$

$$\Leftrightarrow$$

$f(x)$ has a non-removable infinite discontinuity at $x=c$.

Example 11:
Evaluate for each of the following functions: P.S. Don't use a calculator if you don't have to. P.S.S. You don't have to.

I. $h(x)=\ln x$ (a) $h(0)=$ (b) $\lim_{x\to 0^-} h(x)=$ (c) $\lim_{x\to 0^+} h(x)=$ (d) $\lim_{x\to 0} h(x)=$

II. $f(x)=\dfrac{1}{x-3}$ (a) $f(3)=$ (b) $\lim\limits_{x\to 3^-} f(x)=$ (c) $\lim\limits_{x\to 3^+} f(x)=$ (d) $\lim\limits_{x\to 3} f(x)=$

III. $g(x)=\dfrac{1}{(x+2)^2}$ (a) $g(-2)=$ (b) $\lim\limits_{x\to -2^-} g(x)=$ (c) $\lim\limits_{x\to -2^+} g(x)=$ (d) $\lim\limits_{x\to -2} g(x)=$

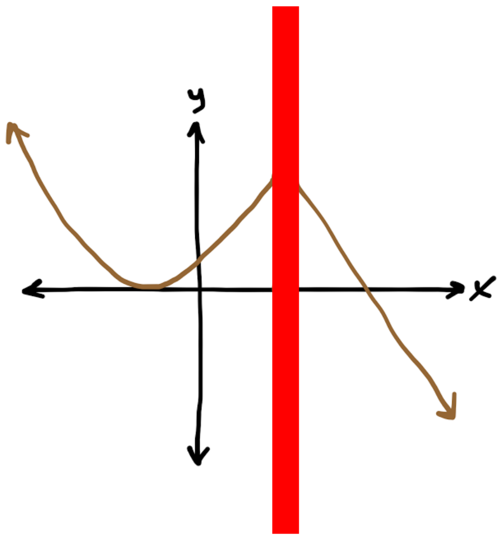
IV. $K(x)=\sec x$ (a) $K\left(\dfrac{\pi}{2}\right)=$ (b) $\lim\limits_{x\to \frac{\pi}{2}^-} K(x)=$ (c) $\lim\limits_{x\to \frac{\pi}{2}^+} K(x)=$ (d) $\lim\limits_{x\to \frac{\pi}{2}} K(x)=$

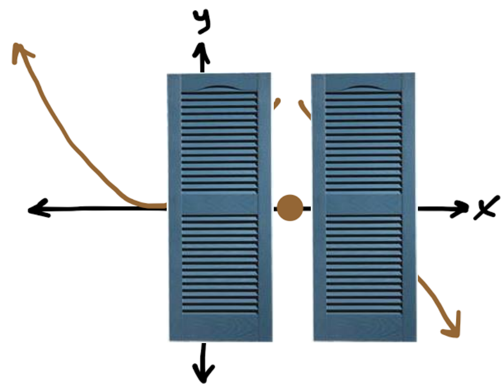
There is one last type of discontinuity that can be a little triggy. It's called an **oscillating discontinuity**.

Example 12:

Examine $f(x)=\sin\left(\dfrac{1}{x}\right)$ in the vicinity of $x=0$. Verify your results by graphing the function and zooming in around $x=0$. What is $\lim\limits_{x\to 0} f(x)$?

Hopefully you're getting comfortable with this new idea of a limit value and how it is categorically different from a function value. Additionally, you've no doubt become aware of how important both the limit value and the function value are important to the idea of continuity at a point.

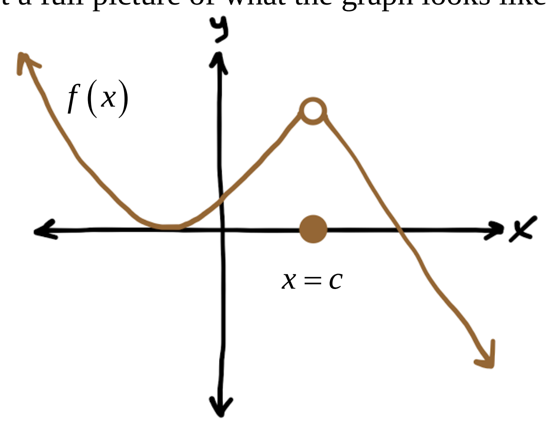




When looking for a limit value at $x = c$, imagine that you've got a thick vertical line covering up $x = c$ with only the graph showing on either side of $x = c$. You are now looking to see what *y*-value(s) the graph is approaching on either side of $x = c$. If the graphs appear to be approaching the same *y*-value, the limit exists and is that *y*-value. Otherwise, the limit does not exist there.

When looking for a function value at $x = c$, imagine that you've got shudders up on either side of $x = c$ with only the vertical sliver at $x = c$ visible between them. You are now looking for the dot or the piece of the graph that exists in that narrow sliver. If it exists, the *y*-value of the dot is the function value $f(c)$.

Put the two together, and you get a full picture of what the graph looks like at $x = c$.



Example 13:
Below is the graph of a function $f(x)$. Evaluate the following:

(a) $\lim_{x \to -2} f(x) =$

(b) $\lim_{x \to 2^-} f(x) =$

(c) $f(4) =$

(d) $\lim_{x \to 4} f(x) =$

(e) $\lim_{x \to 11^+} f(x) =$

(f) $\lim_{x \to 5^-} f(x) =$

(g) $f(2) =$

(h) $\lim_{x \to 6^+} f(x) =$

(i) $\lim_{x \to 6^-} f(x) =$

(j) $\lim_{x \to 8} f(x) =$

(k) $f(-2) =$

(l) $\lim_{x \to 0^-} f(x) =$