Name Key Date__________ Period______

Worksheet 3.2—Rolle's Theorem and the MVT

Show all work. No calculator unless otherwise stated.

Multiple Choice

$D_f: \{x \mid x \le 6\}$

D 1. Determine if the function $f(x) = x\sqrt{6-x}$ satisfies the hypothesis of Rolle's Theorem on the interval $[0,6]$, and if it does, find all numbers c satisfying the conclusion of that theorem.

(A) 2, 3 (B) 4, 5 (C) 5 (D) 4 (E) hypothesis not satisfied

① • f is cont. on $[0,6]$
• f is diff on $(0,6)$
• $f(0) = 0 = f(6)$
So Rolle's Theorem Applies

② $f'(x) = (6-x)^{\frac{1}{2}} + \frac{1}{2}x(6-x)^{-\frac{1}{2}}(-1)$ factor out least factor!

$f'(x) = (6-x)^{-\frac{1}{2}}\left[(6-x) - \frac{1}{2}x\right]$

$= \dfrac{6 - \frac{3}{2}x}{\sqrt{6-x}}$

③ $6 - \frac{3}{2}x = 0$

$\boxed{x = 4}$ ✓ in interval

B 2. Let f be a function defined on $[-1,1]$ such that $\underbrace{f(-1) = f(1)}_{\text{Even function}}$. Consider the following properties that f might have:

I. f is continuous on $[-1,1]$, differentiable on $(-1,1)$.

II. $f(x) = \cos^3 x$

III. $f(x) = |\sin \pi x| = \sqrt{(\sin \pi x)^2}$

Which properties ensure that there exists a c in $(-1,1)$ at which $f'(c) = 0$? (Rolle's Theorem)

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I, II, and III

I. ✓ Rolles Theorem

II. ✓ f is cont on $[-1,1]$, $f'(x) = 3\cos^2 x \sin x$ So diff on $(-1,1)$, $f(-1) = f(1)$

III. X f is cont on $[-1,1]$, $f'(x) = \dfrac{\pi \sin(\pi x)\cos(\pi x)}{|\sin \pi x|} \rightarrow \sin \pi x = 0$, $\pi x = 0$, $x = 0$; $\pi x = 1$, $x = \frac{1}{\pi}$ Not diffa $(-1,1)$

D 3. Determine if the function $f(x) = x^3 - x - 1$ satisfies the hypothesis of the MVT on $[-1,2]$. If it does, find all possible values of c satisfying the conclusion of the MVT. $D_f: \mathbb{R}$

(A) $-\frac{1}{2}$

(B) $-1, 1$

(C) 0

(D) 1

(E) hypothesis not satisfied

① f is cont on $[-1,2]$ + diff on $(-1,2)$ So MVT applies

② $\dfrac{f(2) - f(-1)}{2-(-1)} = f'(x)$

$\dfrac{5-(-1)}{3} = 3x^2 - 1$

$2 = 3x^2 - 1$

$3x^2 = 3$

$x^2 = 1$ $x = \pm 1$

③ Since $x = -1$ not on $(-1,2)$ So $x = 1$

A 4. Determine if the function $f(x) = x + x^{2/3}(1-x)^{1/3}$ satisfies the hypothesis of the MVT on $[0,1]$. If it does, find all possible values of c satisfying the conclusion of the MVT. (You will have to factor out least powers.)

(A) $\frac{2}{3}$

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) $\frac{1}{3}$

(E) hypothesis not satisfied

$D_f: \mathbb{R}$

① • f is cont. on $[0,1]$

• $f'(x) = 1 + \left[\frac{2}{3}x^{-\frac{1}{3}}(1-x)^{\frac{1}{3}} + x^{\frac{2}{3}} \cdot \frac{1}{3}(1-x)^{-2/3}(-1)\right]$

$f'(x) = 1 + \left[\frac{1}{3}x^{-\frac{1}{3}}(1-x)^{-2/3}\right][2(1-x) - x]$

$= 1 + \dfrac{2-3x}{3\sqrt[3]{x} \cdot \sqrt[3]{(1-x)^2}}$ $x \neq 0, x \neq 1$

So f is not diff at $x=0$ & $x=1$ but not on $(0,1)$ so MVT applies

② $2 - 3x = 0$

$x = \frac{2}{3}$

B 5. Which of the following functions below satisfy the hypothesis of the MVT?

I. $f(x) = \dfrac{1}{x+1}$ on $[0,2]$ VA at $x=-1$ Not on $[0,2]$ so continuous on $[0,2]$ $f'(x) = -1(x+1)^{-2} = \frac{-1}{(x+1)^2}$ $x \neq -1$ Not on $(0,2)$

II. $f(x) = x^{1/3}$ on $[0,1]$ cont on $[0,1]$ $f'(x)$ dne at $x=0$, ok on $(0,1)$

III. $f(x) = |x|$ on $[-1,1]$ cont on $[-1,1]$, $f'(x)$ dne at $0 \in (-1,1)$ so NO

(A) I only (B) I and II only (C) I and III only (D) II only (E) II and III only

C 6. As a graduation present, Jenna received a sports car which she drives very fast but very, very smoothly and safely. She always covers the 53 miles from her apartment in Austin, Texas to her parents' home in New Braunfels in less than 48 minutes. To slow her down, her dad decides to change the speed limit (he has connections.) Which one of the speed limits below is the highest speed her father can post, but still catch her speeding at some point on her trip?

(A) 55 mph (B) 70 mph (C) 65 mph (D) 50 mph (E) 60 mph

$\frac{53 \text{ mi}}{48 \text{ min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}}$ Avg vel. 66.25 mph by MVP = inst

B 7. Consider the following statements:

I. $f(x)$ is continuous on $[a,b]$

II. $f(x)$ is differentiable on (a,b)

III. $f(a) = f(b)$ only for Rolle's Theorem

Which of the above statements are required in order to guarantee a $c \in (a,b)$ such that $f'(c)(b-a) = f(b) - f(a)$? $\Rightarrow f'(c) = \dfrac{f(b)-f(a)}{(b-a)}$ MVT

(A) I only (B) I and II only (C) I, II, and III (D) III only (E) I and III only

Short Answer

8. Without looking at your notes, state the Mean Value Theorem.

If... f is continuous of [a,b] and diff on (a,b)

then... there exists $x=c$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

corner

9. Determine if Rolle's Theorem can be applied to the following functions on the given interval. If so, find the value(s) guaranteed by the theorem.

(a) $f(x)=\cos 2x$ on $\left[-\frac{\pi}{12},\frac{\pi}{6}\right]$

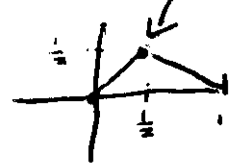
$f\left(\frac{-\pi}{12}\right)=\cos\left(\frac{-\pi}{6}\right)=\frac{\sqrt{3}}{2}$

$f\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

$\frac{\sqrt{3}}{2}\neq\frac{1}{2}$ so

Rolles Theorem does not apply!

(b) $g(x)=\begin{cases} x, & 0\le x\le \frac{1}{2} \\ 1-x, & \frac{1}{2}<x\le 1\end{cases}$ on $[0,1]$



cont:

$\lim_{x\to\frac{1}{2}^-} g(x)=\frac{1}{2}$

$\lim_{x\to\frac{1}{2}^+} g(x)=\frac{1}{2}$

g(x) is continuous on [0,1]

diff:

$\lim_{x\to\frac{1}{2}^-} g'(x)=1$

$\lim_{x\to\frac{1}{2}^+} g'(x)=-1$

$\lim_{x\to\frac{1}{2}^-} g'(x)\neq\lim_{x\to\frac{1}{2}^+} g'(x)$

g'(x) is not diff on (0,1)

so

Rolles Theorem does not apply

10. Determine if the MVT can be applied to the following functions on the given interval. If so, find the exact value(s) guaranteed by the theorem. Be sure to show your set up in finding the value(s).

$D_f=\{x\mid x>1\}$

(a) $f(x)=\ln(x-1)$ on $[2,4]$

- f is cont. on [2,4]
- f is diff on (2,4)

MVT applies so

$\dfrac{f(4)-f(2)}{4-2}=\dfrac{\ln 3-\ln 1}{2}=\frac{1}{2}\ln 3$

$f'(x)=\dfrac{1}{x-1}\Rightarrow\dfrac{1}{x-1}=\frac{1}{2}\ln 3$

$x-1=\frac{2}{\ln 3}$

$\boxed{x=\frac{2}{\ln 3}+1}$

(b) $f(x)=\begin{cases}\arcsin x, & -1\le x<1 \\ \frac{x}{2}, & 1\le x\le 3\end{cases}$ on $[-1,3]$.

continuity:

$\lim_{x\to1^-} f(x)=\frac{\pi}{2}$

$\lim_{x\to1^+} f(x)=\frac{1}{2}$

$\lim_{x\to1^-} f(x)\neq\lim_{x\to1^+} f(x)$

The MVT does not apply since f(x) is not continuous on [-1,3]

(c) $g(x)=\dfrac{x+1}{x}$ on $\left[\frac{1}{2},2\right]$

g(x) is continuous on $[\frac{1}{2},2]$

g(x) is diff on $(\frac{1}{2},2)$

So MVT applies

$\dfrac{g(2)-g(\frac{1}{2})}{2-\frac{1}{2}}=-1$

$g'(x)=\dfrac{x-(x+1)}{x^2}=-\dfrac{1}{x^2}$

$-\dfrac{1}{x^2}=-1\quad x=\pm1\rightarrow\boxed{x=1}$

(d) $f(x)=2\sin x+\sin 2x$ on $[0,\pi]$

f(x) is continuous on $[0,\pi]$

f(x) is diff on $[0,\pi]$

So MVT applies

$\dfrac{f(\pi)-f(0)}{\pi-0}=0$

$f'(x)=2\cos x+2\cos 2x$ (Divide by 2)

$2\cos x+2\cos 2x=0$

$2\cos x+2[2\cos^2 x-1]=0$

$\cos x+2\cos^2 x-1=0$

$2\cos^2 x+\cos x-1=0$

$(2\cos x-1)(\cos x+1)$

$\cos x=+\frac{1}{2}$ $\cos x=-1$

$\boxed{x=\frac{\pi}{3}}$ $x=0, x=\pi$ not on $(0,\pi)$

11. **(Calculator permitted)** For $f(x) = -x^4 + 4x^3 + 8x^2 + 5$ on $[0,5]$

(a) Determine if the MVT can be applied on the given interval. If so, find the value(s) guaranteed by the theorem.

① f is continuous on [0,5]
f is diff. on (0,5) } MVT Applies

② $\frac{f(5)-f(0)}{5-0} = 15$, $f'(x) = -4x^3 + 12x^2 + 16x$

③ $f'(x) = 15$
$x = .674 \rightarrow A$
$x = 3.793 \rightarrow B$

(b) Find the equation of the secant line on $[0,5]$ m = 15 use (0,5) for a pt on f(x)

$y = 15x + 5$

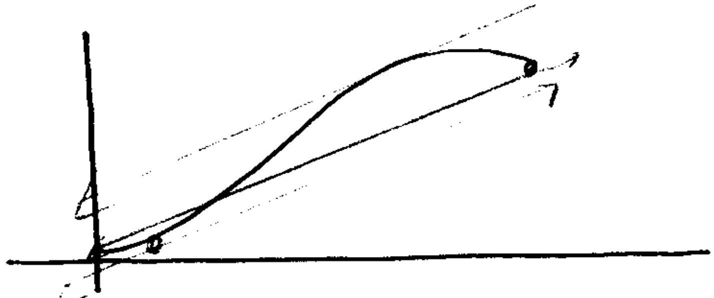
(c) Find the equation of the tangent line at any value of *c* found above.

Use (A, f(A)) OR Use B, f(B)

$y = 9.646 + 15(x - A)$ $y = 131.4 + 15(x - B)$

(d) On your calculator, sketch a graph of $f(x)$ on $[0,5]$ along with the secant and tangent line(s). Sketch the graph below.

Window $[0,5] \rightarrow x$
$[-10,150] \rightarrow y$



12. Let *f* satisfy the hypothesis of Rolle's Theorem on an interval $[a,b]$, such that $f'(c) = 0$. Using your knowledge of transformations, find an interval, in terms of *a* and *b*, for the function *g* over which Rolle's theorem can be applied, and find the corresponding critical value of *g*, in terms of *c*. Assume *k* is a non-zero constant such that $k > 0$.

(a) $g(x) = f(x) + k$ V. Shift K units

New Interval: $[a, b]$

New *x*-value: c

(b) $g(x) = f(x-k)$ H. Shift K units

New Interval: $[a+k, b+k]$

New *x*-value: $c+k$

(c) $g(x) = kf(x)$ V. Dilation by K

New Interval: $[a, b]$

New *x*-value: c

(d) $g(x) = f(kx)$ H dilation by $\frac{1}{k}$

New Interval: $\left[\frac{a}{k}, \frac{b}{k}\right]$

New *x*-value: $\frac{c}{k}$

$\neg D \to C$ on $(0,1)$
but needs to be cont on $[0,1]$
So...

13. The function $f(x)=\begin{cases}0, & x=0\\ 1-x, & 0<x\le 1\end{cases}$ is differentiable on $(0,1)$ and satisfies $f(0)=f(1)$. However, its derivative is never zero on $(0,1)$. Does this contradict the Mean Value Theorem? Explain why or why not.

Continuous?
$\lim_{x\to 0^-} f(x)=0$
$\lim_{x\to 0^+} f(x)=1$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, $f(x)$ is not continuous on $[0,1]$
$\therefore$ MVT Does not apply

14. Determine the values of a, b, and c such that the function f satisfies the hypothesis of the MVT on the interval $[0,3]$.

$$f(x)=\begin{cases}1, & x=0\\ ax+b, & 0<x\le 1\\ x^2+4x+c, & 1<x\le 3\end{cases}$$

$f'(x)=\begin{cases}a & 0<x\le 1\\ 2x+4 & 1<x\le 3\end{cases}$

1 Contin. at $x=0$
$\lim_{x\to 0^-} f(x)=1$
$\lim_{x\to 0^+} f(x)=b$
$\boxed{b=1}$

2 Cont at $x=1$
$\lim_{x\to 1^-} f(x)=a+b$
$\lim_{x\to 1^+} f(x)=5+c$
$a+b=5+c$
$a=4+c$

3 Diff at $x=1$
$\lim_{x\to 1^-} f'(x)=a$
$\lim_{x\to 1^+} f'(x)=6$
$\boxed{a=6}$

4 $a=4+c$
$6=4+c$
$\boxed{c=2}$

15. Suppose that we know that $f(x)$ is continuous and differentiable on $[6,15]$. Let's also suppose that we know that $f(6)=-2$ and that $f'(x)\le 10$ for all $x\in[6,15]$. What is the largest possible value for $f(15)$?

$\frac{f(15)-f(6)}{15-6}\le 10$

$\frac{f(15)-(-2)}{9}\le 10$

$f(15)+2\le 90$

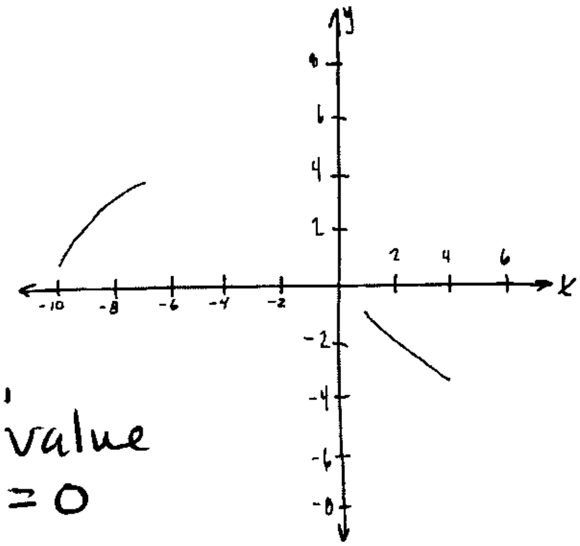
$f(15)\le 88$

16. Let $f(x)=\tan x$. Show that $f(\pi)=f(2\pi)$ but that there is not number $c\in(\pi,2\pi)$ such that $f'(c)=0$. Why does this not contradict Rolle's Theorem?

$f(\pi)=0$
$f(2\pi)=0$

f is not continuous on $[\pi, 2\pi]$
$\therefore$ it does not contradict Rolle's Theorem

17. The figure at right shows two **parts** of the graph of a function $f(x)$ that is continuous on $[-10,4]$ and differentiable on $(-10,4)$. It so happens that the derivative $f'(x)$ is also continuous on $[-10,4]$.



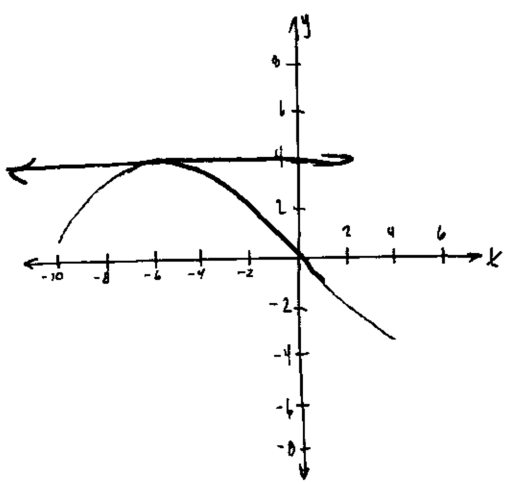
(a) Explain why f must have at least one zero in $[-10,4]$.

Since f is continuous on $[-10,4]$ and since $f(4) < 0 < f(-10)$, by the IVT there exists a value $c \in (-10,4)$ such that $f(c) = 0$

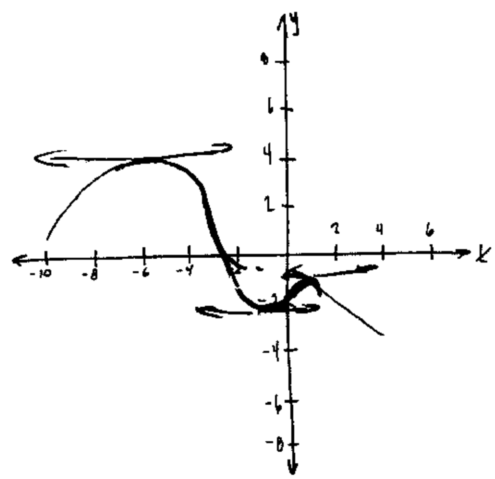
(b) Explain why f' must also have at least one zero in the interval $[-10,4]$. What are these zeros called?

Since f is continuous on $[-10,4]$ and diff on $(-10,4)$ and since $f'(x)$ is cont. on $[-10,4]$ $f'(x) > 0$ on $(-10,-6)$, $f'(x) < 0$ $(2,4)$ then $f'(c) = 0$ for c $(-10,4)$, That value is called a critical value

(c) Make a possible sketch of the function with **one** zero of f' on the interval $[-10,4]$.



(d) Make a **possible** sketch of the function with **at least two** zeros of f' on the interval $[-10,4]$.



(e) Were the conditions of continuity of f and f' necessary to do parts (a) through (d)? Explain.

Yes, the conditions were necessary for a & b.

These conditions were not necessary for c + d